Closed Loop Control of a Type 2 Single Pole Sysem

Define the plant and controller, calculate the closed loop gains.

$$G_a(s) = \frac{K}{s^2}$$

Type 2 system

$$G_c(s) = Kp + Kd \cdot s$$

A simple PD controller.

$$CLTF(s) = \frac{(Kp + Kd \cdot s) \cdot \frac{K}{s^2}}{1 + (Kp + Kd \cdot s) \cdot \frac{K}{s^2}}$$

Closed loop transfer function (CTLF) for combined actuator and controller transfer functions.

$$CLTF(s) = \frac{G_c(s) \cdot G_a(s)}{1 + G_c(s) \cdot G_a(s)}$$

Closed loop transfer function with the characteristic equation in the denominator.

$$CLTF(s) = \frac{(Kp + Kd \cdot s) \cdot K}{s^2 + K \cdot Kp + K \cdot Kd \cdot s}$$

$$(s + \lambda)^2 \text{ expand } \rightarrow s^2 + 2 \cdot s \cdot \lambda + \lambda^2$$

Desired characteristic equation. In this case a critically damped response is desired.

Equate the powers of s for the coefficients of the characteristic equation of the closed loop transfer function and the desired characteristic equation. The solve for the two unknown gains using the two equations. This is simple enough to do by inspection.

Given

$$\lambda^2 = K \cdot Kp$$

Match the coefficients for the 0 power of s

$$2 \cdot \lambda = K \cdot Kd$$

Match the coefficients for s

$$Find(Kp, Kd) \rightarrow \begin{pmatrix} \frac{\lambda^2}{K} \\ 2 \cdot \frac{\lambda}{K} \end{pmatrix}$$

These are the symbolic formulas for the proportional and derivative gains.
This is sooo simple!

Closed Loop Control of a Type 2 Single Pole Sysem

Define System Parameters

$K := 10$ Millimeters per second squared per % control output

Define the desired response moving the pole on the negative real axis.
Place two closed loop poles at -λ.

$\lambda := 2 \cdot \pi \cdot 10$ $\lambda = 62.832$

Calculate the Controller Gains Using the Symbolic Formulas

$Kp := \frac{\lambda^2}{K}$ $Kp = 394.784$

$Kd := 2 \cdot \frac{\lambda}{K}$ $Kd = 12.566$

Simulation

S domain state space arrays

$A_c := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $A_c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$B_c := \begin{pmatrix} 0 \\ K \end{pmatrix}$ $B_c = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$

$C := \begin{pmatrix} 1 & 0 \end{pmatrix}$ $D := 0$

$I := \text{identity}(2)$ $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The formula for converting state space to a Laplace transform transfer function

$C \cdot (s \cdot I - A_c)^{-1} \cdot B_c + D \rightarrow \frac{10}{s^2}$ This should be the plant transfer function if the state space arrays are set up right.

$T := 0.001$ Update interval

Calculate arrays for use in discrete time.

$A := I + \sum_{n=1}^{7} \frac{(A_c \cdot T)^n}{n!}$ $B := \left[I + \sum_{n=1}^{7} \frac{(A_c \cdot T)^n}{(n+1)!} \right] \cdot B_c \cdot T$

$A = \begin{pmatrix} 1 & 0.001 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 5 \times 10^{-6} \\ 0.01 \end{pmatrix}$

Closed Loop Control of a Type 2 Single Pole Sysem

PD Control

$$x_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Initial state

$$u_0 := 0$$

Initial control output

$$PD(r, x, u) := \left| \begin{array}{l} x \leftarrow A \cdot x + B \cdot u \\ u \leftarrow \max[\min[(Kp \;\; Kd) \cdot (r - x), 100], -100] \\ \begin{pmatrix} x \\ u \end{pmatrix} \end{array} \right.$$

$$N := \frac{1}{T} \qquad n := 0..N$$

Simulate 1 second

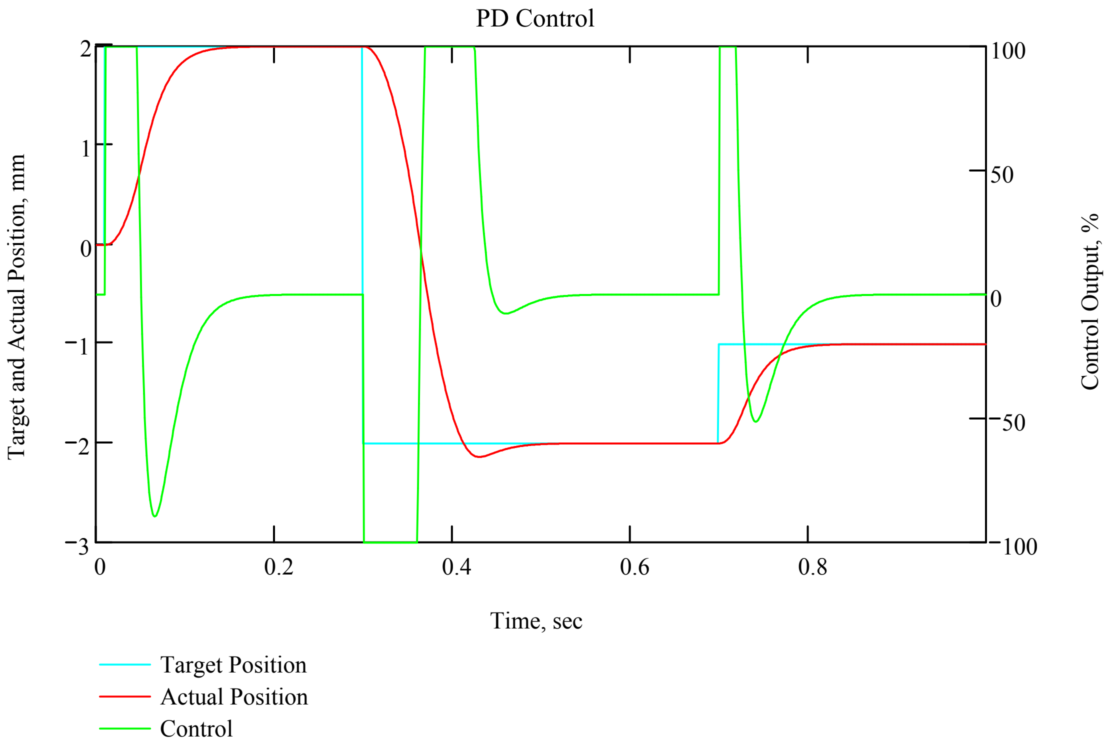
$$r_n := \left| \begin{array}{ll} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } n < 0.01 \cdot N \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \text{if } 0.01 \cdot N \le n \wedge n < 0.3 \cdot N \\ \begin{pmatrix} -2 \\ 0 \end{pmatrix} & \text{if } 0.3 \cdot N \le n \wedge n < 0.7 \cdot N \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \text{otherwise} \end{array} \right.$$

Calculate the target positions and velocities. In this case the target velocities are 0 since that target positions is making step jumps.

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \end{pmatrix} := PD\left(r_n, x_n, u_n\right)$$

Compute the system response and control output

Verify By Simulating and Graphing the Results.



The closed loop poles can only be placed so far to the left on the negative real axis without having the control output saturate. However, if a smooth ramp were generated there wouldn't be a big error to cause saturation.

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PD Control of a Constant Acceleration Ramp

$acc := 500$

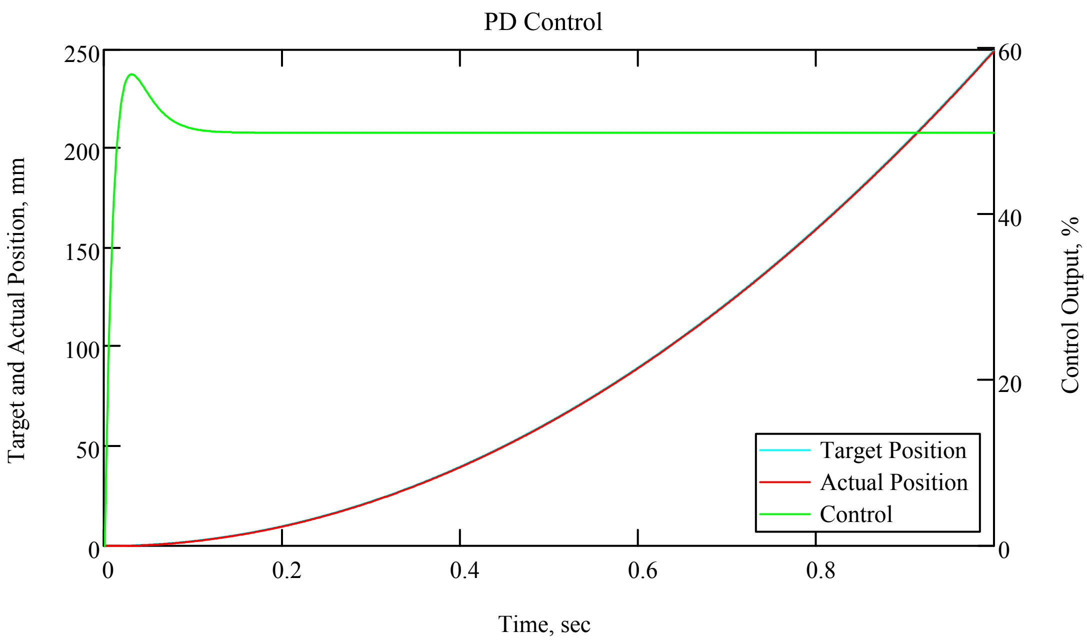
Accelerate at 500mm/sec^2

$$r_n := \begin{bmatrix} \frac{acc}{2} \cdot (n \cdot T)^2 \\ acc \cdot n \cdot T \end{bmatrix}$$

Calculate the target positions and velocities for a constant acceleration ramp.

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \end{pmatrix} := PD\left(r_n, x_n, u_n\right)$$

Compute the system response and control output



Closed Loop Control of a Type 2 Single Pole Sysem

PD Control of a Sine Wave

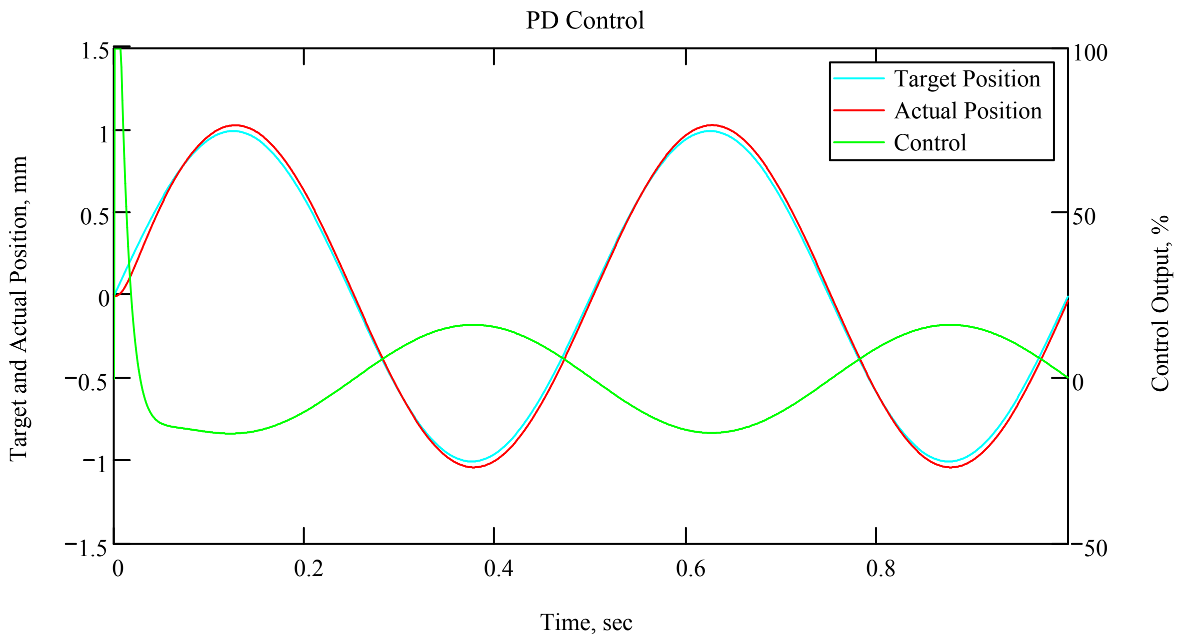
$Amp := 1 \qquad Hz := 2$

$$r_n := \begin{pmatrix} Amp \cdot \sin(2 \cdot \pi \cdot Hz \cdot n \cdot T) \\ 2 \cdot \pi \cdot Hz \cdot Amp \cdot \cos(2 \cdot \pi \cdot Hz \cdot n \cdot T) \end{pmatrix}$$

Calculate the target position and velocities for a sine wave.

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \end{pmatrix} := PD(r_n, x_n, u_n)$$

Compute the system response and control output



The control output looks like it is 180 degrees out of phase with the positions.

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Closed Loop Bode Plot Calculations

$$\mathrm{CLTF}(s) := \frac{(Kp + Kd \cdot s) \cdot \dfrac{K}{s^2}}{1 + (Kp + Kd \cdot s) \cdot \dfrac{K}{s^2}}$$

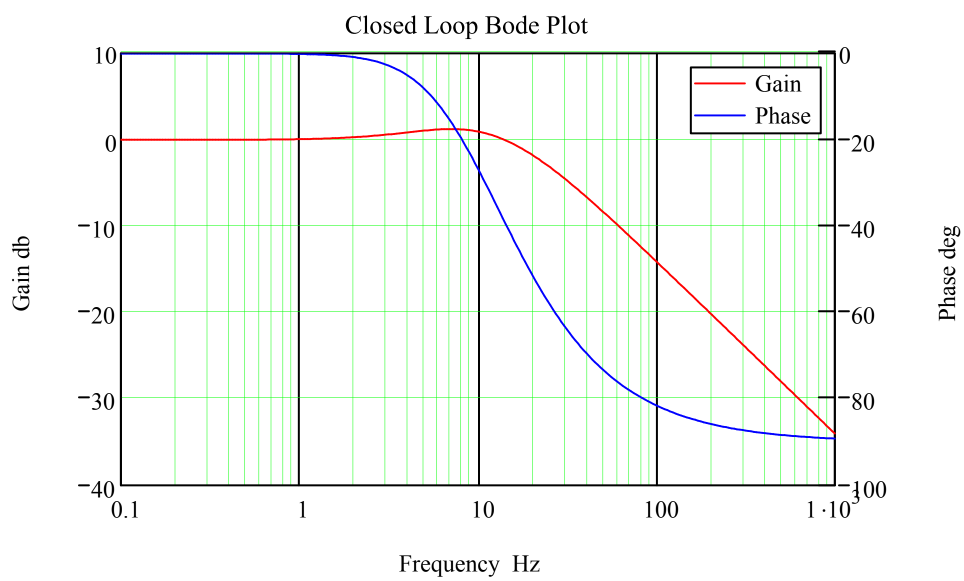
$n := 0..256$

Iterate the frequency from 0.1 HZ to 1000 HZ using 64 steps per decade for a total of 256 iterations.

$$hz_n := 10^{\frac{n}{64}-1}$$

$$\begin{pmatrix} M_n \\ \phi_n \end{pmatrix} := \left| \begin{array}{l} r \leftarrow \mathrm{CLTF}\left[j \cdot (2 \cdot \pi) \cdot hz_n\right] \\ a \leftarrow \dfrac{\arg(r)}{\deg} \\ a \leftarrow a - 360 \ \text{ if } \ n > 0 \wedge \left|a - \phi_{n-1}\right| > 100 \\ \begin{pmatrix} 20 \cdot \log(|r|) \\ a \end{pmatrix} \end{array} \right.$$

Closed loop magnitude of transfer function as a function of frequency



The increase in gain at 10 Hz is the result of the zero. If the derivative gain is in the feed back path only the gain will not rise about 0 db but the phase lag will increase.

Closed Loop Control of a Type 2 Single Pole Sysem

Graph Closed Loop Poles and Zeros

$$CLTF := \left[\frac{(Kp + Kd \cdot s) \cdot \frac{K}{s^2}}{1 + (Kp + Kd \cdot s) \cdot \frac{K}{s^2}} \left|\begin{matrix}\text{expand} \\ \text{explicit}\end{matrix}\right. \rightarrow \frac{K \cdot Kp + K \cdot Kd \cdot s}{s^2 + K \cdot Kp + K \cdot Kd \cdot s}\right]$$

$$d := \text{denom}(CLTF) \text{ coeffs}, s \rightarrow \begin{pmatrix} K \cdot Kp \\ K \cdot Kd \\ 1 \end{pmatrix}$$

$$n := \text{numer}(CLTF) \text{ coeffs}, s \rightarrow \begin{pmatrix} K \cdot Kp \\ K \cdot Kd \end{pmatrix}$$

$$poles := \text{polyroots}(d)$$

$$zeros := \text{polyroots}(n)$$

$$poles = \begin{pmatrix} -62.832 \\ -62.832 \end{pmatrix}$$

$$zeros = -31.416$$