PID Position Control of a Small Motor System

PID Control of a Type 1 Single Pole System
Define System Parameters

$K := 0.20$ — Open loop gain. mm/s per DAC count

$\tau := 0.05$ — Open loop time constant in seconds. This is pretty slow.

$$G_p(s) := \frac{K}{s \cdot (\tau \cdot s + 1)}$$

Open loop transfer function

Calculate PID gains using Ackermann's method. The desired response should have no over shoot so place the three poles on the negative real axis in the s domain.

Continuous state space arrays for calculating PID gains. The system transition matrix should be 3x 3 to calculate 3 gains for the PID.

$T := 0.001$ — Update interval. 1Khz

$$A_c := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\tau} \end{pmatrix} \qquad B_c := \begin{pmatrix} 0 \\ 0 \\ \frac{K}{\tau} \end{pmatrix} \qquad I := \text{identity}(\text{rows}(B_c))$$

Calculate arrays for use in discrete time.

$$A := I + \sum_{n=1}^{7} \frac{(A_c \cdot T)^n}{n!} \qquad B := \left[I + \sum_{n=1}^{7} \frac{(A_c \cdot T)^n}{(n+1)!} \right] \cdot B_c \cdot T$$

$$A = \begin{pmatrix} 1 & 0.001 & 4.966833 \times 10^{-7} \\ 0 & 1 & 0.00099 \\ 0 & 0 & 0.980199 \end{pmatrix} \qquad B = \begin{pmatrix} 6.633466 \times 10^{-10} \\ 1.986733 \times 10^{-6} \\ 0.00396 \end{pmatrix}$$

$$\lambda_z := \exp\left(-\frac{2 \cdot T}{\tau}\right) \qquad \lambda_z = 0.961$$

λ_z are the pole locations in the z domain. It should be on the positive real axis.

$$\alpha_c := (A - \lambda_z \cdot I)^3$$

The next few step solve Ackermann's equation to compute PID gains.

$$\alpha_c = \begin{pmatrix} 0.00006 & 4.612404 \times 10^{-6} & 9.818729 \times 10^{-8} \\ 0 & 0.00006 & 2.648659 \times 10^{-6} \\ 0 & 0 & 7.311815 \times 10^{-6} \end{pmatrix}$$

Finally calculate the PID gains
Calculating the PID gains symbolically is takes fewer calculations and is much more accurate once the symbolic formulas for the PID gains is found.

$$(Ki \;\; Kp \;\; Kd) := (0 \;\; 0 \;\; 1) \cdot \text{augment}(B, A \cdot B, A^2 \cdot B)^{-1} \cdot \alpha_c$$

$$(Ki \;\; Kp \;\; Kd) = (1.522 \times 10^4 \;\; 1.149 \times 10^3 \;\; 24.124)$$

Simulation

Define the continuous time state space arrays

$$\alpha := \frac{1}{\tau}$$

$$A_c := \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} \qquad A_c = \begin{pmatrix} 0 & 1 \\ 0 & -20 \end{pmatrix}$$

$$B_c := \begin{pmatrix} 0 \\ K \cdot \alpha \end{pmatrix} \qquad B_c = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$I := \mathrm{identity}(\mathrm{rows}(B_c))$$ 2x2 identity matrix

Calculate arrays for use in discrete time.

$$A := I + \sum_{n=1}^{7} \frac{A_c{}^n \cdot T^n}{n!} \qquad B := \left(I + \sum_{n=1}^{7} \frac{A_c{}^n \cdot T^n}{(n+1)!} \right) \cdot B_c \cdot T$$

$$A = \begin{pmatrix} 1 & 0.00099 \\ 0 & 0.980199 \end{pmatrix} \qquad B = \begin{pmatrix} 1.986733 \times 10^{-6} \\ 0.00396 \end{pmatrix}$$

PID Control

$x_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \text{Position} \\ \text{Velocity} \end{pmatrix}$ — Initial state.

$u_0 := 0$ — Initial control output

$u_{i_0} := 0$ — Initial integrator contribution to the control output

$$\mathrm{PID}(r, x, u, u_i) := \left| \begin{array}{l} x \leftarrow A \cdot x + B \cdot u \\ u_{pd} \leftarrow (Kp \;\; Kd) \cdot (r - x) \\ u_i \leftarrow \max\left[\min\left[u_i + Ki \cdot T \cdot \left(r_0 - x_0\right), 32767 - u_{pd}\right], -32767 - u_{pd}\right] \\ \begin{pmatrix} x \\ u_i + u_{pd} \\ u_i \end{pmatrix} \end{array} \right.$$

$N := \dfrac{4}{T} \qquad n := 0..N$ — Simulate 4 seconds

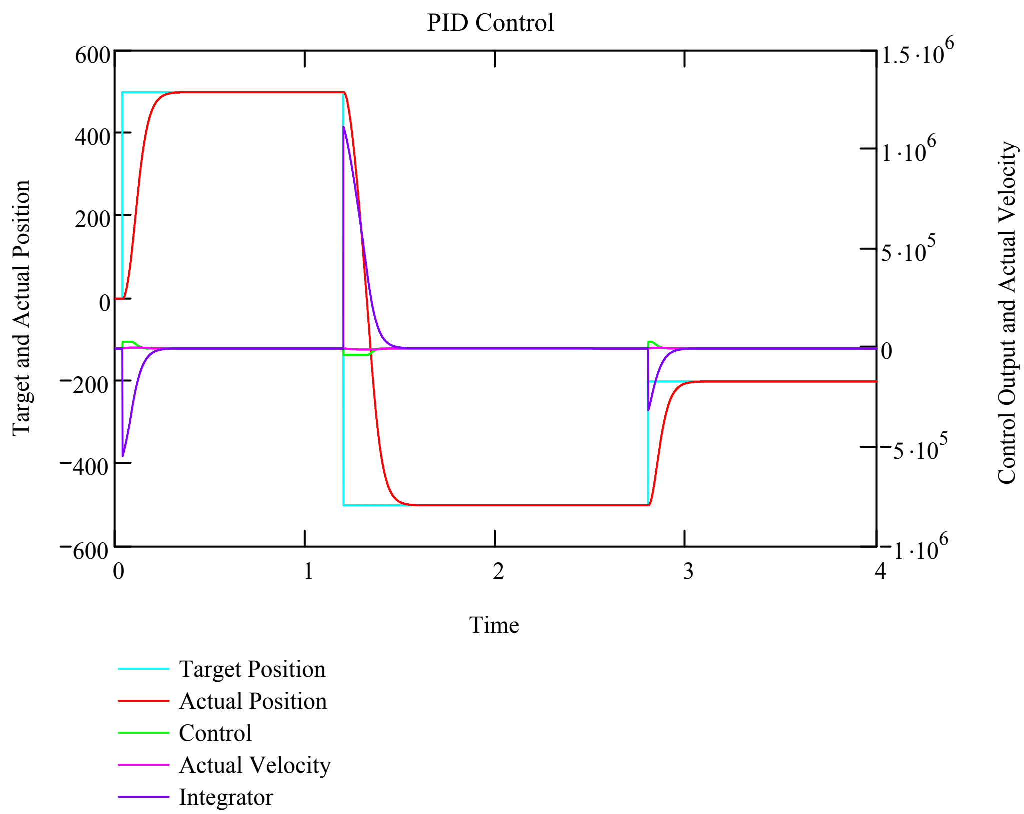
$$r_n := \left| \begin{array}{ll} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } n < 0.01 \cdot N \\ \begin{pmatrix} 500 \\ 0 \end{pmatrix} & \text{if } 0.01 \cdot N \le n \wedge n < 0.3 \cdot N \\ \begin{pmatrix} -500 \\ 0 \end{pmatrix} & \text{if } 0.3 \cdot N \le n \wedge n < 0.7 \cdot N \\ \begin{pmatrix} -200 \\ 0 \end{pmatrix} & \text{otherwise} \end{array} \right.$$

Calculate the target positions and velocities. In this case the target velocities are 0 since that target positions is making step jumps.

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \\ u_{i_{n+1}} \end{pmatrix} := \mathrm{PID}\left(r_n, x_n, u_n, u_{i_n}\right)$$

Compute the system response and control output

PID Position Control of a Small Motor System



Following a Sine Wave

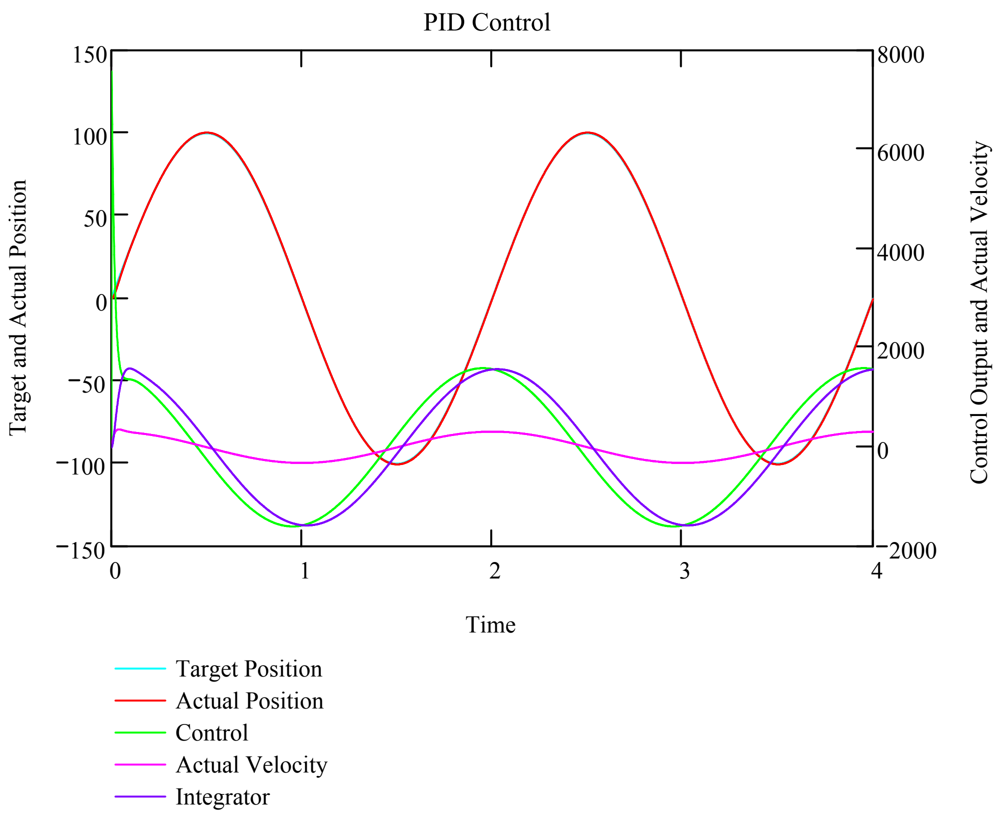
Note, there is no feed forward

$Amp := 100$ mm

$$r_n := Amp \cdot \begin{pmatrix} \sin(\pi \cdot n \cdot T) \\ \cos(\pi \cdot n \cdot T) \cdot \pi \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \\ u_{i_{n+1}} \end{pmatrix} := PID\left(r_n, x_n, u_n, u_{i_n}\right)$$

Compute the system response and control output



Bode Plot Calculations For PID

$$T(s) := \frac{\left(\frac{Ki}{s} + Kp + Kd \cdot s\right) \cdot \frac{K \cdot \alpha}{s \cdot (s + \alpha)}}{1 + \left(\frac{Ki}{s} + Kp + Kd \cdot s\right) \cdot \frac{K \cdot \alpha}{s \cdot (s + \alpha)}}$$

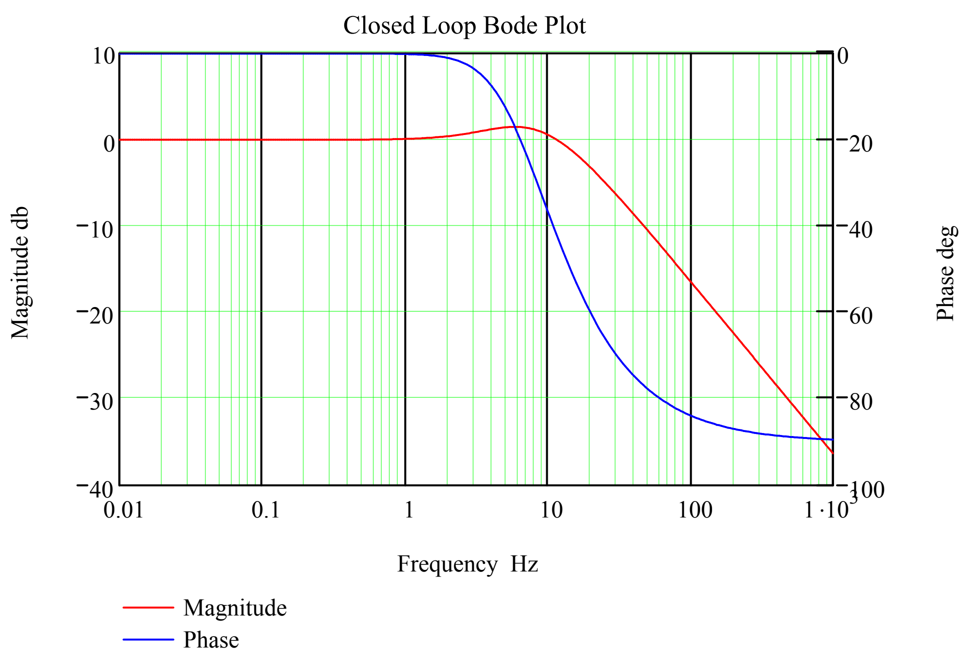
$n := 0 .. 320$

Iterate the frequency from .01 HZ to 1000 HZ using 64 steps per decade for a total of 320 iterations.

$$hz_n := 10^{\frac{n}{64} - 2}$$

$$\begin{pmatrix} M_n \\ \phi_n \end{pmatrix} := \left| \begin{array}{l} r \leftarrow T\left[j \cdot (2 \cdot \pi) \cdot hz_n\right] \\ a \leftarrow \frac{\arg(r)}{deg} \\ a \leftarrow a - 360 \ \text{ if } \ n > 0 \wedge \left|a - \phi_{n-1}\right| > 100 \\ \begin{pmatrix} 20 \cdot \log(|r|) \\ a \end{pmatrix} \end{array} \right.$$

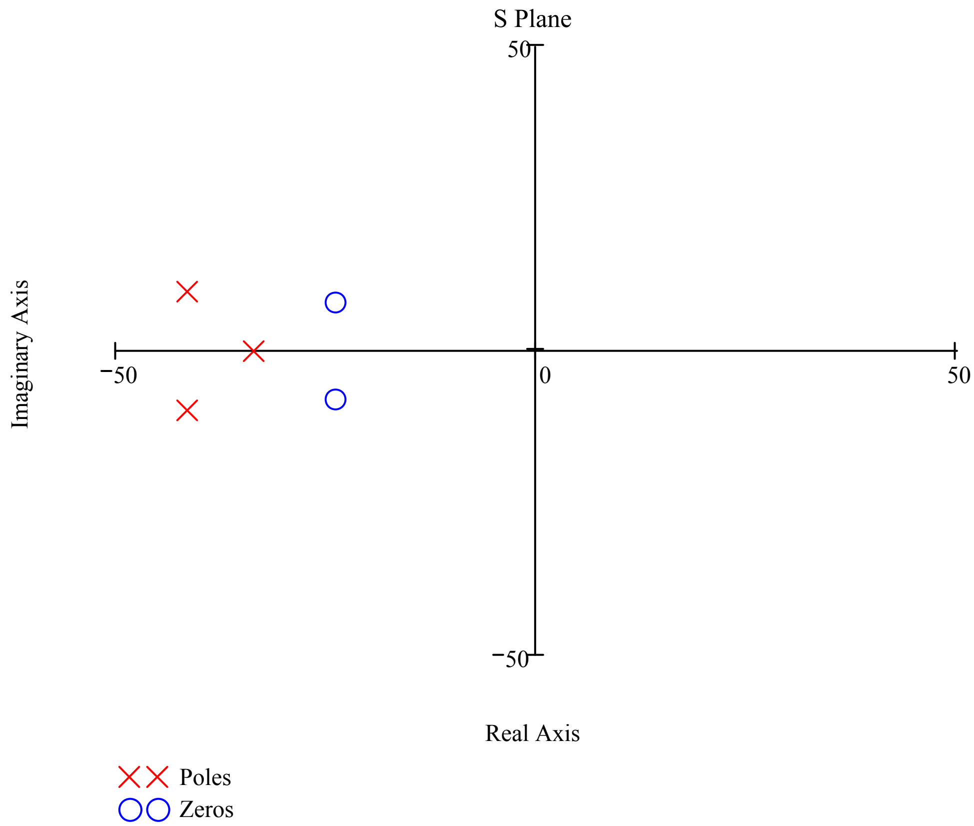
Calculate magnitude and phase for T(s)



Graph Poles and Zeros for PID

$$poles := polyroots\left(\begin{pmatrix} K \cdot \alpha \cdot Ki \\ K \cdot \alpha \cdot Kp \\ K \cdot \alpha \cdot Kd + \alpha \\ 1 \end{pmatrix}\right) \qquad poles = \begin{pmatrix} -41.471 - 9.74i \\ -41.471 + 9.74i \\ -33.554 \end{pmatrix}$$

$$zeros := polyroots\left(\begin{pmatrix} Ki \\ Kp \\ Kd \end{pmatrix}\right) \qquad zeros = \begin{pmatrix} -23.823 - 7.966i \\ -23.823 + 7.966i \end{pmatrix}$$



PI-D Control

$x_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \text{Position} \\ \text{Velocity} \end{pmatrix}$

Initial state.

$u_0 := 0$

Initial control output

$u_{i_0} := 0$

Initial integrator contribution to the control output

$$\mathrm{PI\_D}(r, x, u, u_i) := \left| \begin{array}{l} x \leftarrow A \cdot x + B \cdot u \\ u_{pd} \leftarrow (Kp \quad Kd) \cdot \left[\begin{pmatrix} r_0 \\ 0 \end{pmatrix} - x \right] \\ u_i \leftarrow \max\left[\min\left[u_i + Ki \cdot T \cdot \left(r_0 - x_0\right), 32767 - u_{pd}\right], -32767 - u_{pd}\right] \\ \begin{pmatrix} x \\ u_i + u_{pd} \\ u_i \end{pmatrix} \end{array} \right.$$

$N := \frac{4}{T} \qquad n := 0 .. N$

Simulate 4 seconds

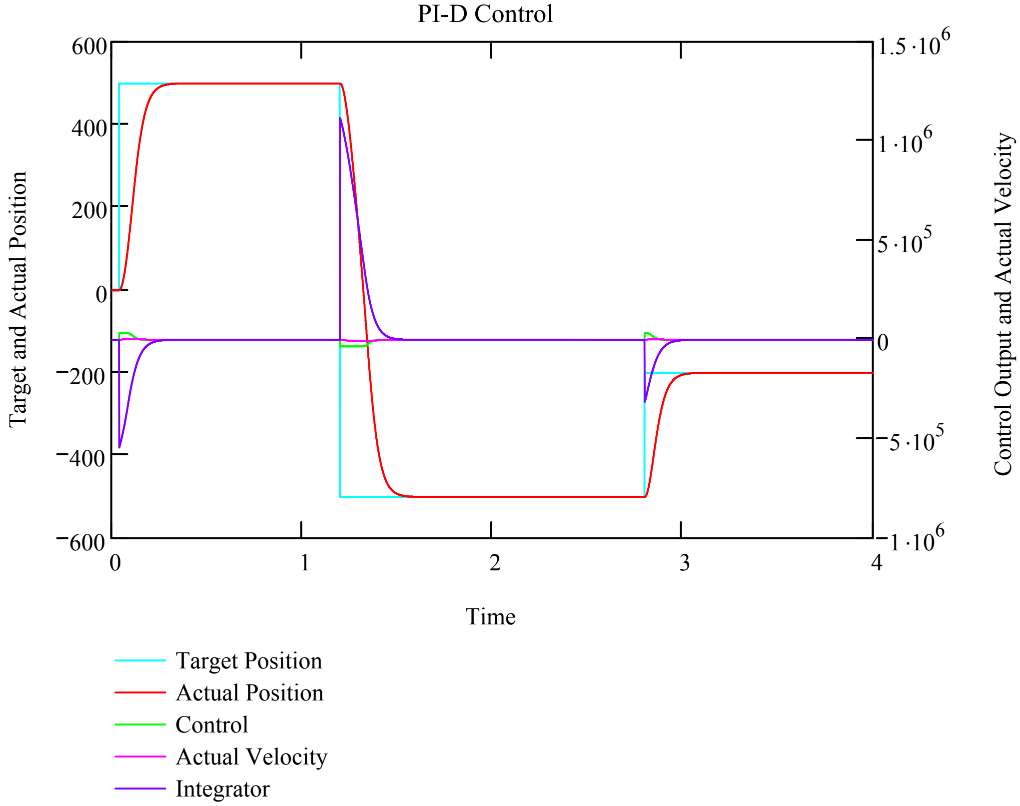
$$r_n := \left| \begin{array}{ll} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } n < 0.01 \cdot N \\ \begin{pmatrix} 500 \\ 0 \end{pmatrix} & \text{if } 0.01 \cdot N \le n \wedge n < 0.3 \cdot N \\ \begin{pmatrix} -500 \\ 0 \end{pmatrix} & \text{if } 0.3 \cdot N \le n \wedge n < 0.7 \cdot N \\ \begin{pmatrix} -200 \\ 0 \end{pmatrix} & \text{otherwise} \end{array} \right.$$

Calculate the target positions and velocities. In this case the target velocities are 0 since that target positions is making step jumps.

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \\ u_{i_{n+1}} \end{pmatrix} := \mathrm{PID}\left(r_n, x_n, u_n, u_{i_n}\right)$$

Compute the system response and control output

PID Position Control of a Small Motor System



PID Position Control of a Small Motor System

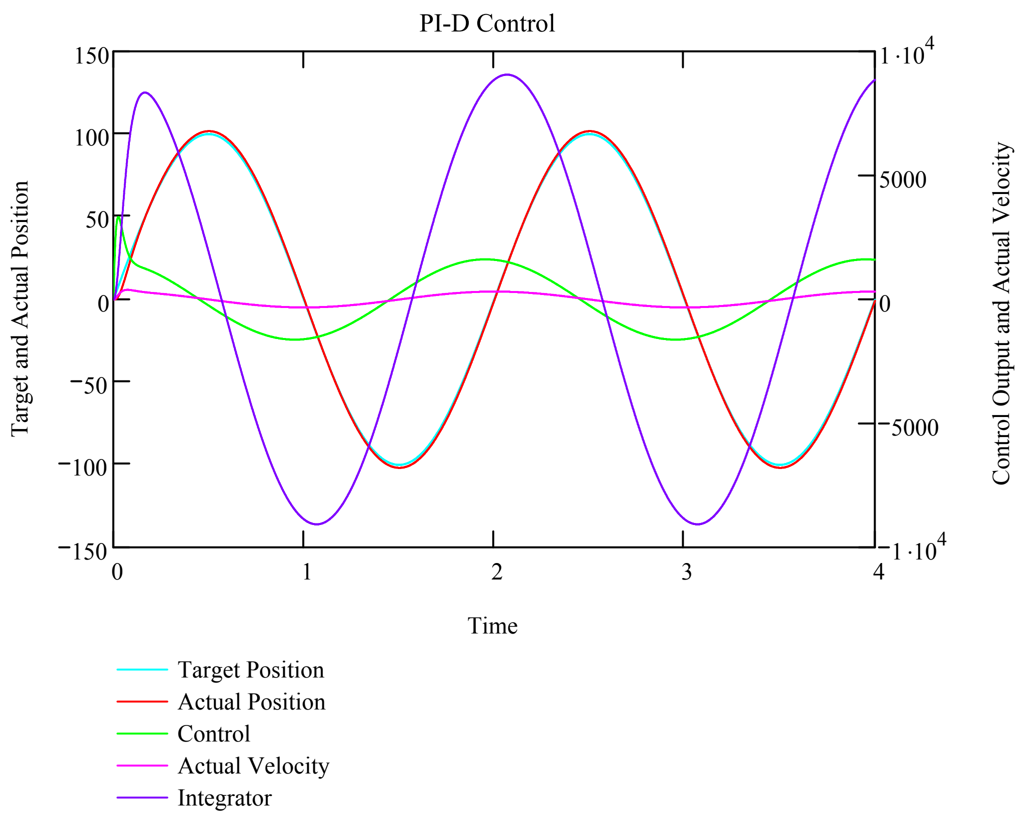
Following a Sine Wave

Note, there is no feed forward

$$r_n := Amp \cdot \begin{pmatrix} \sin(\pi \cdot n \cdot T) \\ \cos(\pi \cdot n \cdot T) \cdot \pi \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \\ u_{i_{n+1}} \end{pmatrix} := PI\_D\left(r_n, x_n, u_n, u_{i_n}\right)$$

Compute the system response and control output



Bode Plot Calculations For PI-D

$$T(s) := \frac{\left(\frac{Ki}{s} + Kp\right) \cdot \frac{K \cdot \alpha}{s \cdot (s + \alpha)}}{1 + \left(\frac{Ki}{s} + Kp + Kd \cdot s\right) \cdot \frac{K \cdot \alpha}{s \cdot (s + \alpha)}}$$

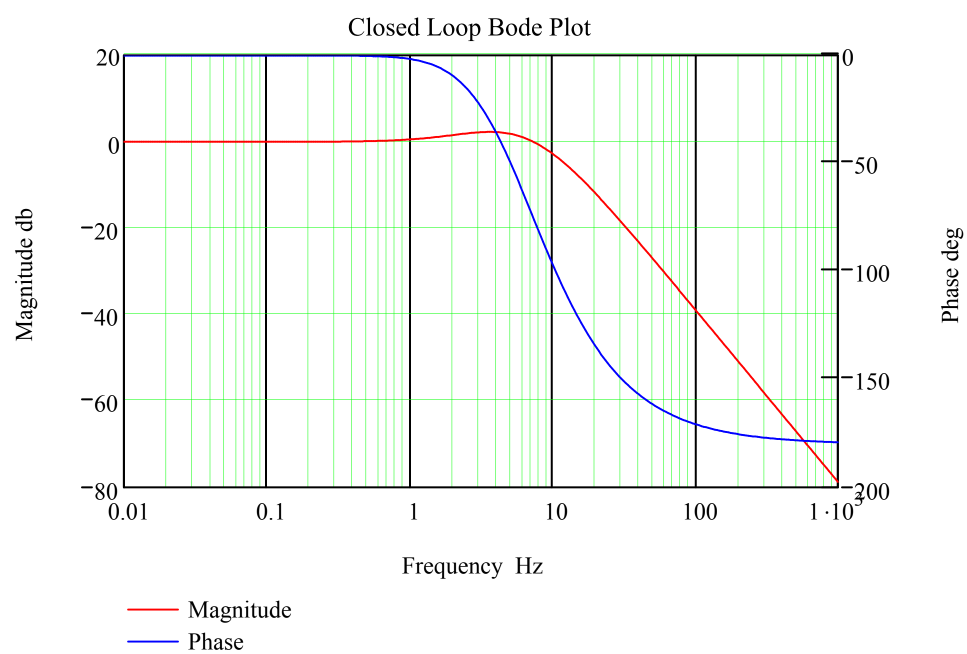
$n := 0 .. 320$

Iterate the frequency from .01 HZ to 1000 HZ using 64 steps per decade for a total of 320 iterations.

$$hz_n := 10^{\frac{n}{64} - 2}$$

$$\begin{pmatrix} M_n \\ \phi_n \end{pmatrix} := \left| \begin{array}{l} r \leftarrow T\left[j \cdot (2 \cdot \pi) \cdot hz_n\right] \\ a \leftarrow \frac{\arg(r)}{deg} \\ a \leftarrow a - 360 \quad \text{if } n > 0 \wedge \left|a - \phi_{n-1}\right| > 100 \\ \begin{pmatrix} 20 \cdot \log(|r|) \\ a \end{pmatrix} \end{array} \right.$$

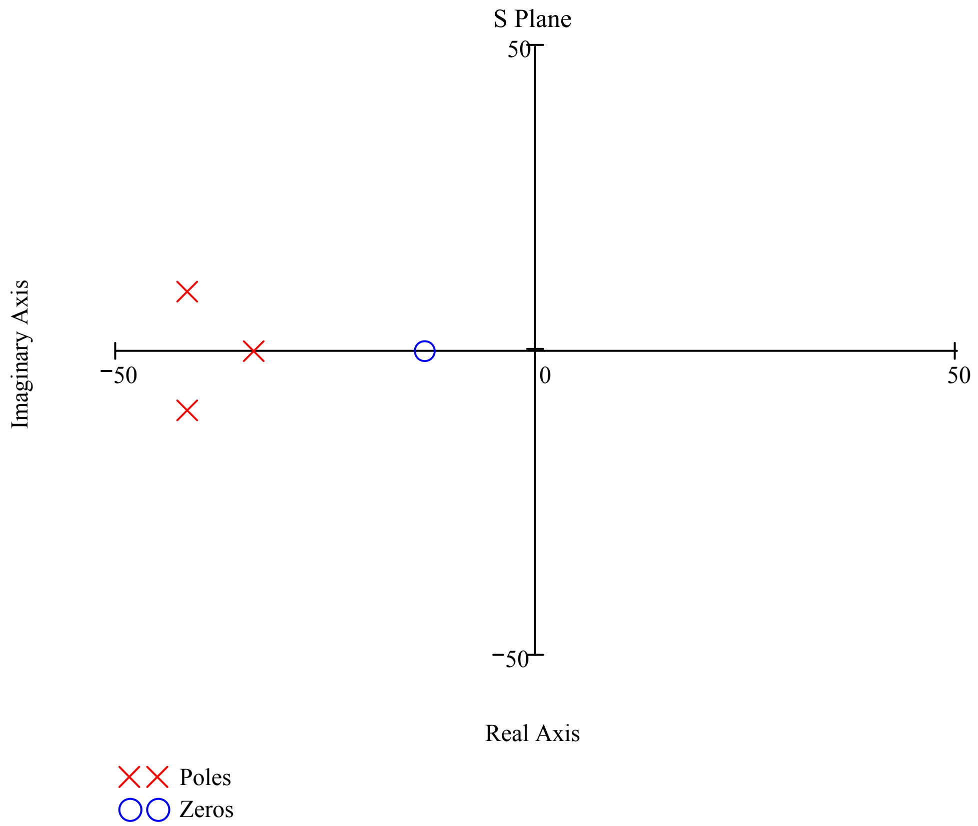
Calculate magnitude and phase for T(s)



Graph Poles and Zeros for PI-D

$$poles := \text{polyroots}\left(\begin{pmatrix} K \cdot \alpha \cdot Ki \\ K \cdot \alpha \cdot Kp \\ K \cdot \alpha \cdot Kd + \alpha \\ 1 \end{pmatrix}\right) \qquad poles = \begin{pmatrix} -41.471 - 9.74i \\ -41.471 + 9.74i \\ -33.554 \end{pmatrix}$$

$$zeros := \text{polyroots}\left(\begin{pmatrix} Ki \\ Kp \end{pmatrix}\right) \qquad zeros = -13.244$$



I-PD Control

$x_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \text{Position} \\ \text{Velocity} \end{pmatrix}$ — Initial state.

$u_0 := 0$ — Initial control output

$u_{i_0} := 0$ — Initial integrator contribution to the control output

$$IPD(r, x, u, u_i) := \left| \begin{array}{l} x \leftarrow A \cdot x + B \cdot u \\ u_{pd} \leftarrow (Kp \;\; Kd) \cdot -x \\ u_i \leftarrow \max\left[\min\left[u_i + Ki \cdot T \cdot \left(r_0 - x_0\right), 32767 - u_{pd}\right], -32767 - u_{pd}\right] \\ \begin{pmatrix} x \\ u_i + u_{pd} \\ u_i \end{pmatrix} \end{array} \right.$$

$N := \frac{4}{T} \qquad n := 0..N$ — Simulate 4 seconds

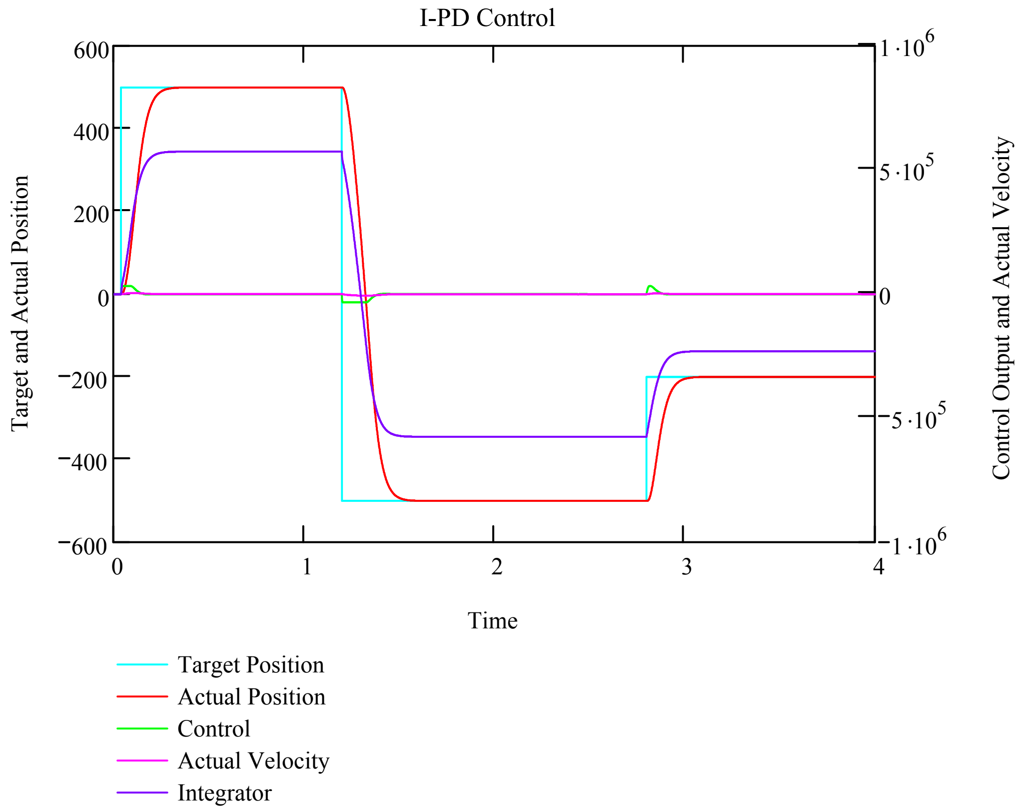
$$r_n := \left| \begin{array}{ll} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } n < 0.01 \cdot N \\ \begin{pmatrix} 500 \\ 0 \end{pmatrix} & \text{if } 0.01 \cdot N \le n \wedge n < 0.3 \cdot N \\ \begin{pmatrix} -500 \\ 0 \end{pmatrix} & \text{if } 0.3 \cdot N \le n \wedge n < 0.7 \cdot N \\ \begin{pmatrix} -200 \\ 0 \end{pmatrix} & \text{otherwise} \end{array} \right.$$

Calculate the target positions and velocities. In this case the target velocities are 0 since that target positions is making step jumps.

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \\ u_{i_{n+1}} \end{pmatrix} := IPD\left(r_n, x_n, u_n, u_{i_n}\right)$$

Compute the system response and control output

PID Position Control of a Small Motor System



Following a Sine Wave

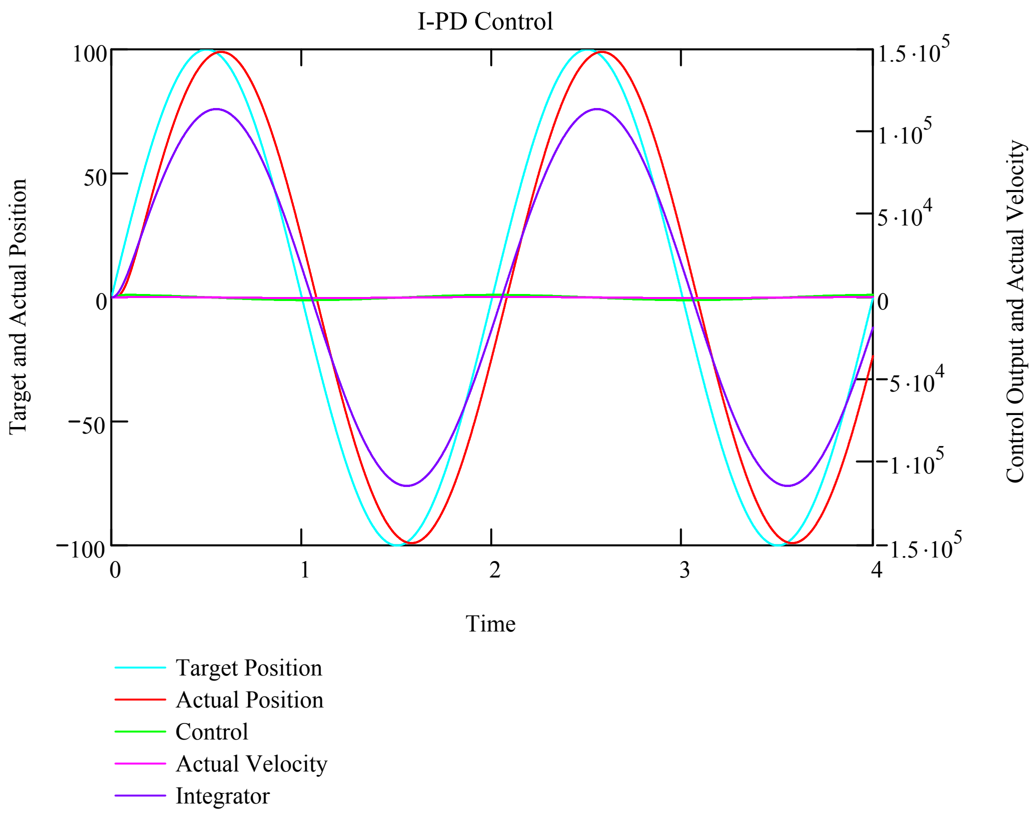
Note, there is no feed forward

$Amp := 100$ mm

$$r_n := Amp \cdot \begin{pmatrix} \sin(\pi \cdot n \cdot T) \\ \cos(\pi \cdot n \cdot T) \cdot \pi \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \\ u_{i_{n+1}} \end{pmatrix} := IPD\left(r_n, x_n, u_n, u_{i_n}\right)$$

Compute the system response and control output



Bode Plot Calculations For I-PD

$$T(s) := \frac{\left(\frac{Ki}{s}\right) \cdot \frac{K \cdot \alpha}{s \cdot (s + \alpha)}}{1 + \left(\frac{Ki}{s} + Kp + Kd \cdot s\right) \cdot \frac{K \cdot \alpha}{s \cdot (s + \alpha)}}$$

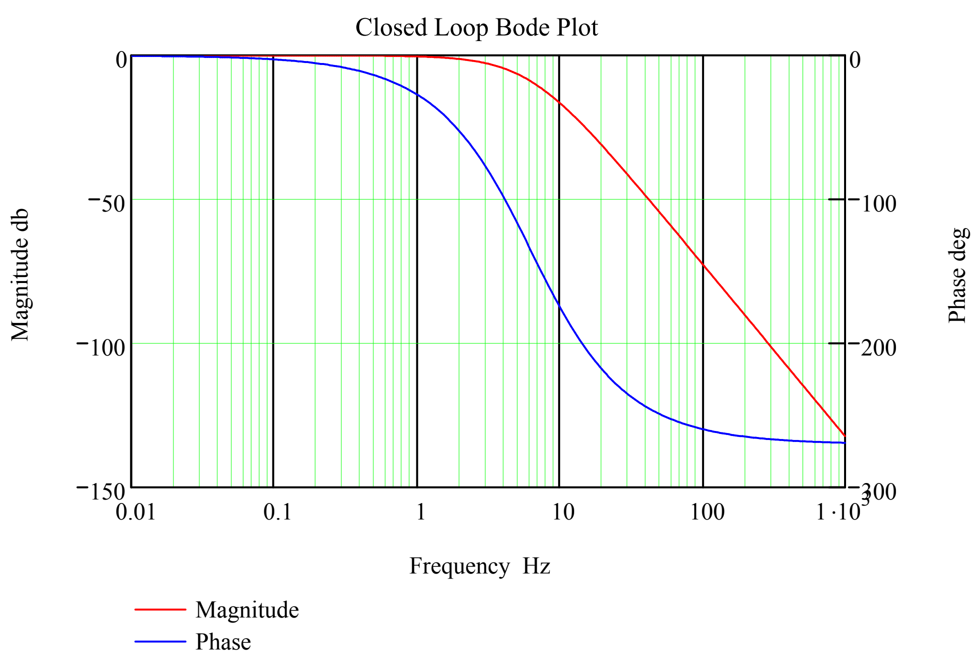
$n := 0 .. 320$

Iterate the frequency from .01 HZ to 1000 HZ using 64 steps per decade for a total of 320 iterations.

$$hz_n := 10^{\frac{n}{64} - 2}$$

$$\begin{pmatrix} M_n \\ \phi_n \end{pmatrix} := \left| \begin{array}{l} r \leftarrow T\left[j \cdot (2 \cdot \pi) \cdot hz_n\right] \\ a \leftarrow \frac{\arg(r)}{deg} \\ a \leftarrow a - 360 \ \text{ if } \ n > 0 \wedge \left|a - \phi_{n-1}\right| > 100 \\ \begin{pmatrix} 20 \cdot \log(|r|) \\ a \end{pmatrix} \end{array} \right.$$

Calculate magnitude and phase for T(s)



Graph Poles and Zeros for I-PD

$$\text{poles} := \text{polyroots}\left(\begin{pmatrix} K \cdot \alpha \cdot Ki \\ K \cdot \alpha \cdot Kp \\ K \cdot \alpha \cdot Kd + \alpha \\ 1 \end{pmatrix}\right) \qquad \text{poles} = \begin{pmatrix} -41.471 - 9.74i \\ -41.471 + 9.74i \\ -33.554 \end{pmatrix}$$