PID Control of a Type 2 Integrator

Calculate PID Controller Gains For A Simple Type 2 System

$$G_a(s) = \frac{K}{s^2}$$

A simple type 2 system

$$G_c(s) = Kp + \frac{Ki}{s} + Kd \cdot s$$

Standard PID controller

$$CLTF(s) = \frac{\left(Kp + \frac{Ki}{s} + Kd \cdot s\right) \cdot \frac{K}{s^2}}{1 + \left(Kp + \frac{Ki}{s} + Kd \cdot s\right) \cdot \frac{K}{s^2}}$$

Closed loop transfer function for controller and second order system.

$$CLTF(s) = \frac{G_c(s) \cdot G_a(s)}{1 + G_c(s) \cdot G_a(s)}$$

$$CLTF(s) = \left(Ki + Kp \cdot s + Kd \cdot s^2\right) \cdot \frac{K}{s^3 + K \cdot Ki + K \cdot Kp \cdot s + K \cdot Kd \cdot s^2}$$

$$CLTF(s) = \frac{\left(Ki + Kp \cdot s + Kd \cdot s^2\right) \cdot K}{s^3 + K \cdot Ki + K \cdot Kp \cdot s + K \cdot Kd \cdot s^2}$$

Simplify the CLTF and collect on powers of s

$$(s + \lambda)^3$$

Desired characteristic equation with three real poles at -λ

$$s^3 + 3 \cdot s^2 \cdot \lambda + 3 \cdot s \cdot \lambda^2 + \lambda^3$$

The desired characteristic equation in powers of s.

Given

$$\lambda^3 = K \cdot Ki$$

$$3 \cdot \lambda^2 = K \cdot Kp$$

$$3 \cdot \lambda = K \cdot Kd$$

Solve for Ki, Kp, and Kd by finding the values of Ki, Kp and Kd that will provide the desired characteristic equation. This time it is trivial

$$Find(Ki, Kp, Kd) \rightarrow \begin{pmatrix} \frac{\lambda^3}{K} \\ 3 \cdot \frac{\lambda^2}{K} \\ 3 \cdot \frac{\lambda}{K} \end{pmatrix}$$

The symbobolic formulas for the PID gains.

PID Control of a Type 2 Integrator

Define System Parameters

$K := 10$ Millimeters per second squared per % control output

Define the desired response moving the pole on the negative real axis.
Place three closed loop poles at -λ.

$\lambda := 2 \cdot \pi \cdot 10$ $\lambda = 62.832$

Calculate the Controller Gains Using the Symbolic Formulas

$Ki := \frac{\lambda^3}{K}$ $Ki = 24805.021$

$Kp := 3 \cdot \frac{\lambda^2}{K}$ $Kp = 1184.353$

$Kd := 3 \cdot \frac{\lambda}{K}$ $Kd = 18.85$

State Space Simulation

$A_c := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $A_c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$B_c := \begin{pmatrix} 0 \\ K \end{pmatrix}$ $B_c = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$

$C := \begin{pmatrix} 1 & 0 \end{pmatrix}$ $D := 0$

$I := \text{identity}(2)$ $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The formula for converting state space to a Laplace transform transfer function

$C \cdot (s \cdot I - A_c)^{-1} \cdot B_c + D \rightarrow \frac{10}{s^2}$ This should be the plant transfer function if the state space arrays are set up right.

$T := 0.001$ Update interval

Calculate arrays for use in discrete time.

$A := I + \sum_{n=1}^{7} \frac{(A_c \cdot T)^n}{n!}$ $B := \left[I + \sum_{n=1}^{7} \frac{(A_c \cdot T)^n}{(n+1)!} \right] \cdot B_c \cdot T$

$A = \begin{pmatrix} 1 & 0.001 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 5 \times 10^{-6} \\ 0.01 \end{pmatrix}$

PID Control of a Type 2 Integrator

PID Control

$x_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ Initial state. Position and Velocity

$u_0 := 0$ Initial control output %

$u_{i_0} := 0$ Initial integrator output%

$$\mathrm{PID}(r, x, u, u_i) := \left| \begin{array}{l} x1 \leftarrow A \cdot x + B \cdot u \\ u_{pd} \leftarrow (Kp \;\; Kd) \cdot (r - x) \\ u_i \leftarrow \max\left[\min\left[u_i + Ki \cdot T \cdot (r - x)_0, 100 - u_{pd}\right], -100 - u_{pd}\right] \\ \begin{pmatrix} x1 \\ u_i + u_{pd} \\ u_i \end{pmatrix} \end{array} \right.$$

$N := \frac{1}{T}$ $n := 0..N$ Simulate 1 second

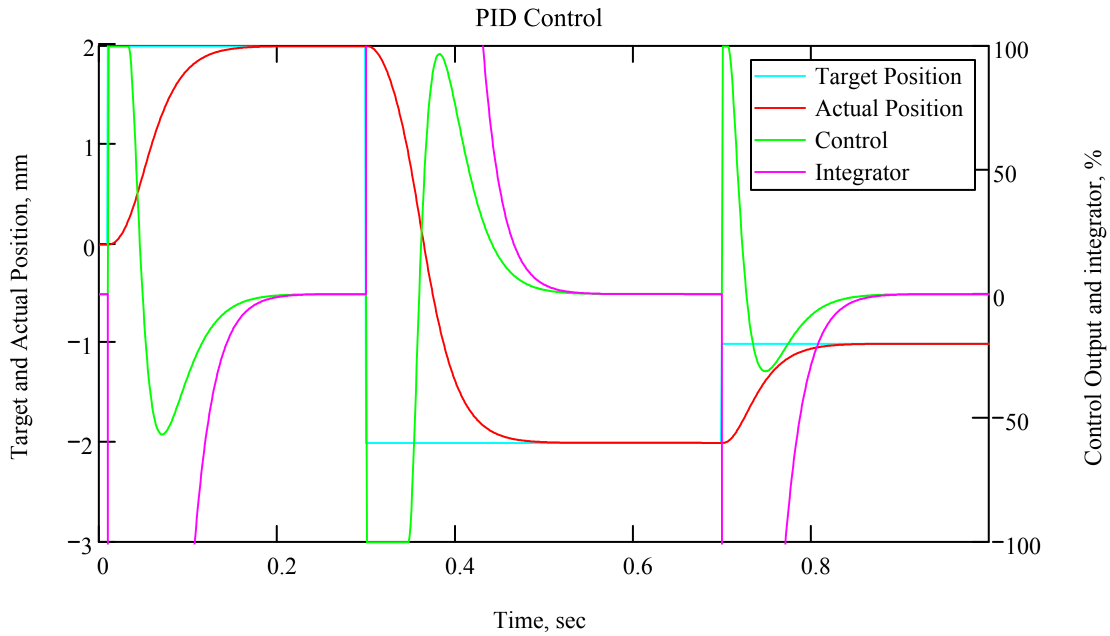
$$r_n := \left\| \begin{array}{ll} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } n < 0.01 \cdot N \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \text{if } 0.01 \cdot N \le n \wedge n < 0.3 \cdot N \\ \begin{pmatrix} -2 \\ 0 \end{pmatrix} & \text{if } 0.3 \cdot N \le n \wedge n < 0.7 \cdot N \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \text{otherwise} \end{array} \right.$$

Calculate the target positions and velocities. In this case the target velocities are 0 since that target positions is making step jumps.

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \\ u_{i_{n+1}} \end{pmatrix} := \mathrm{PID}(r_n, x_n, u_n, u_{i_n})$$

Compute the system response and control output

PID Control of a Type 2 Integrator



The closed loop poles can only be placed so far to the left on the negative real axis without having the control output saturate. However, if a smooth ramp were generated there wouldn't be a big error to cause saturation.

PID Control of a Type 2 Integrator

PID Control of a Constant Acceleration Ramp

$acc := 500$

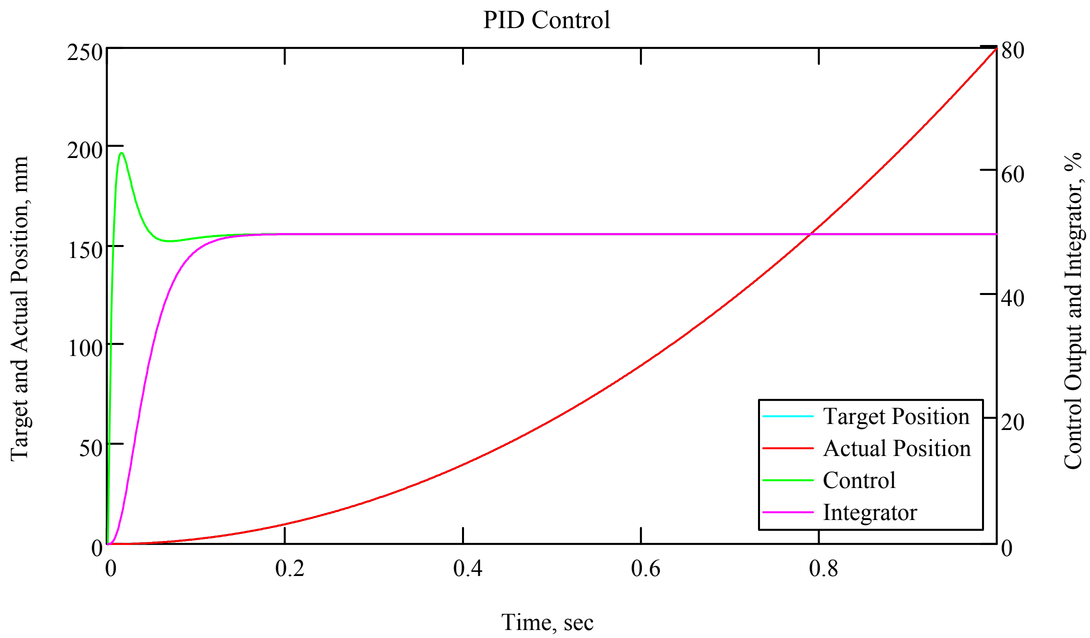
Accelerate at 500mm/sec^2

$$r_n := \begin{bmatrix} \frac{acc}{2}\cdot(n\cdot T)^2 \\ acc\cdot n\cdot T \end{bmatrix}$$

Calculate the target positions and velocities for a constant acceleration ramp.

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \\ u_{i_{n+1}} \end{pmatrix} := PID\left(r_n, x_n, u_n, u_{i_n}\right)$$

Compute the system response and control output



Normally the target positions change smoothly so the control output doesn't saturate.

PID Control of a Type 2 Integrator

PID Control of a Sine Wave

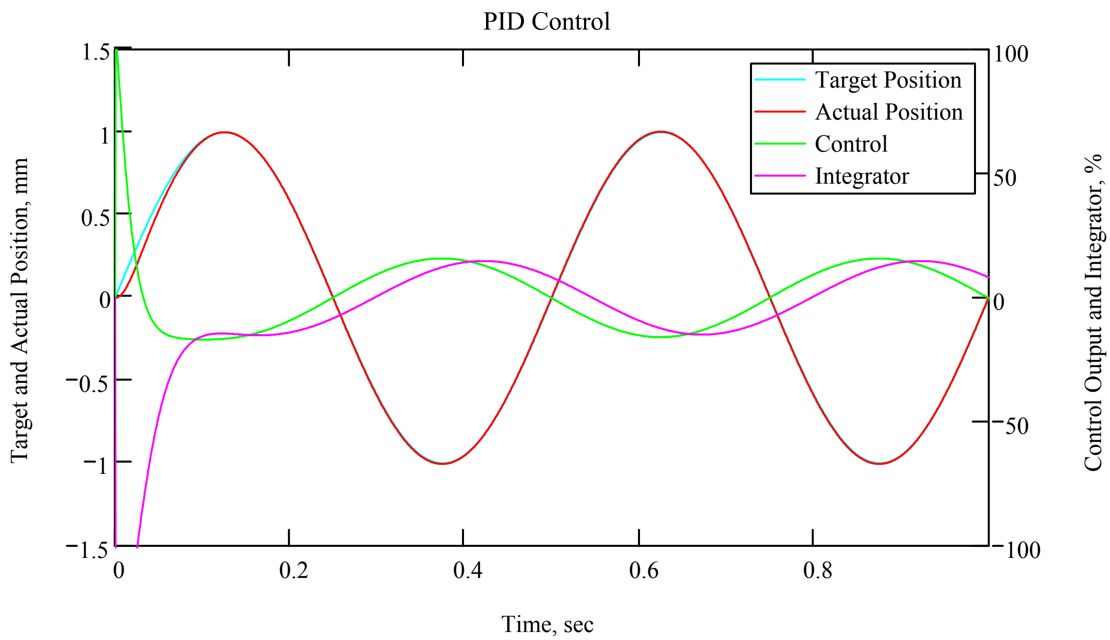
$Amp := 1 \qquad Hz := 2$

$$r_n := \begin{pmatrix} Amp \cdot \sin(2 \cdot \pi \cdot Hz \cdot n \cdot T) \\ 2 \cdot \pi \cdot Hz \cdot Amp \cdot \cos(2 \cdot \pi \cdot Hz \cdot n \cdot T) \end{pmatrix}$$

Calculate the target position and velocities for a sine wave.

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \\ u_{i_{n+1}} \end{pmatrix} := PID\left(r_n, x_n, u_n, u_{i_n}\right)$$

Compute the system response and control output



The control output looks like it is 180 degrees out of phase with the positions.

PID Control of a Type 2 Integrator

Closed Loop Bode Plot Calculations

$$\mathrm{CLTF}(s) := \frac{\left(Kp + \frac{Ki}{s} + Kd\cdot s\right)\cdot\frac{K}{s^2}}{1 + \left(Kp + \frac{Ki}{s} + Kd\cdot s\right)\cdot\frac{K}{s^2}}$$

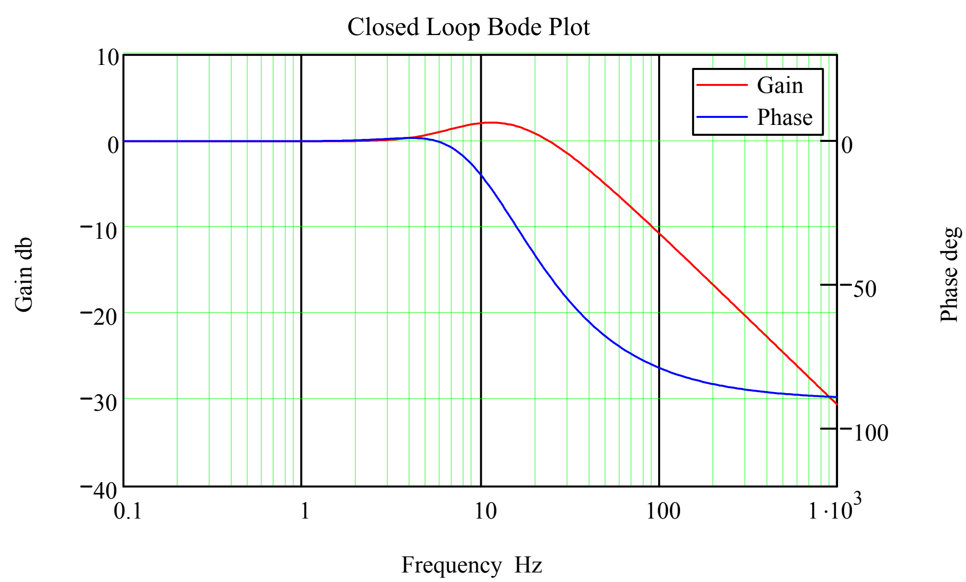
$n := 0..256$

Iterate the frequency from 0.1 HZ to 1000 HZ using 64 steps per decade for a total of 256 iterations.

$$hz_n := 10^{\frac{n}{64}-1}$$

$$\begin{pmatrix} M_n \\ \phi_n \end{pmatrix} := \left| \begin{array}{l} r \leftarrow \mathrm{CLTF}\left[j\cdot(2\cdot\pi)\cdot hz_n\right] \\ a \leftarrow \frac{\arg(r)}{deg} \\ a \leftarrow a - 360 \quad \text{if } n > 0 \wedge \left|a - \phi_{n-1}\right| > 100 \\ \begin{pmatrix} 20\cdot\log(|r|) \\ a \end{pmatrix} \end{array} \right.$$

Closed loop magnitude of transfer function as a function of frequency



The increase in gain at 10 Hz is the result of the zero. If the derivative gain is in the feed back path only the gain will not rise about 0 db but the phase lag will increase.

PID Control of a Type 2 Integrator

Graph Closed Loop Poles and Zeros

$$\text{CLTF} := \left[\frac{\left(Kp + \frac{Ki}{s} + Kd\cdot s\right)\cdot\frac{K}{s^2}}{1 + \left(Kp + \frac{Ki}{s} + Kd\cdot s\right)\cdot\frac{K}{s^2}} \;\Bigg|\begin{matrix}\text{expand}\\ \text{explicit}\end{matrix} \rightarrow \frac{K\cdot Kp\cdot s + K\cdot Ki + K\cdot Kd\cdot s^2}{s^3 + K\cdot Kp\cdot s + K\cdot Ki + K\cdot Kd\cdot s^2}\right]$$

$$d := \text{denom}(\text{CLTF})\ \text{coeffs}, s \rightarrow \begin{pmatrix} K\cdot Ki \\ K\cdot Kp \\ K\cdot Kd \\ 1 \end{pmatrix}$$

$$n := \text{numer}(\text{CLTF})\ \text{coeffs}, s \rightarrow \begin{pmatrix} K\cdot Ki \\ K\cdot Kp \\ K\cdot Kd \end{pmatrix}$$

$$\text{poles} := \text{polyroots}(d)$$

$$\text{zeros} := \text{polyroots}(n)$$

$$\text{poles} = \begin{pmatrix} -62.832 \\ -62.832 \\ -62.832 \end{pmatrix}$$

$$\text{zeros} = \begin{pmatrix} -31.416 + 18.138i \\ -31.416 - 18.138i \end{pmatrix}$$