DC Motor Speed Control

The original problem is on these web pages.
http://www.engin.umich.edu/group/ctm/examples/motor/PID2.html
http://www.engin.umich.edu/group/ctm/examples/motor2/motor.html

$$G_m(s) = \frac{\omega(s)}{V(s)} = \frac{K}{(J\cdot s + b)\cdot(L\cdot s + R) + K^2}$$

Transfer function for a DC Motor

$$G_c(s) = \frac{Ki}{s} + Kp + Kd\cdot s$$

A simple PID controller

$$T(s) = \frac{\left(\frac{Ki}{s} + Kp + Kd\cdot s\right)\cdot\frac{K}{(J\cdot s + b)\cdot(L\cdot s + R) + K^2}}{1 + \left(\frac{Ki}{s} + Kp + Kd\cdot s\right)\cdot\frac{K}{(J\cdot s + b)\cdot(L\cdot s + R) + K^2}}$$

Closed loop transfer function

$$T(s) = \frac{G_c(s)\cdot G_m(s)}{1 + G_c(s)\cdot G_m(s)}$$

$$T(s) = \frac{\left(Ki + Kp\cdot s + Kd\cdot s^2\right)\cdot K}{J\cdot s^3\cdot L + (J\cdot R + b\cdot L + K\cdot Kd)\cdot s^2 + \left(K^2 + K\cdot Kp + b\cdot R\right)\cdot s + K\cdot Ki}$$

$$T(s) = \frac{\frac{\left(Ki+Kp\cdot s+Kd\cdot s^2\right)\cdot K}{J\cdot L}}{s^3 + \frac{J\cdot R + b\cdot L + K\cdot Kd}{J\cdot L}\cdot s^2 + \frac{K^2 + K\cdot Kp + b\cdot R}{J\cdot L}\cdot s + \frac{K\cdot Ki}{J\cdot L}}$$

$$(s + \lambda)^3$$

Desired characteristic equation. In this case a critically damped response is desired with 3 poles at $-\lambda$.

Find the difference between the system characteristic equation and the desired characteristic equation and the solve for the gains.

$$DiffCE := s^3 + \frac{J\cdot R + b\cdot L + K\cdot Kd}{J\cdot L}\cdot s^2 + \frac{K^2 + K\cdot Kp + b\cdot R}{J\cdot L}\cdot s + \frac{K\cdot Ki}{J\cdot L} - (s + \lambda)^3$$

$$DiffCE \left| \begin{matrix} coeffs, s \\ solve, \begin{pmatrix} Ki \\ Kp \\ Kd \end{pmatrix} \end{matrix} \right. \rightarrow \left[\lambda^3\cdot J\cdot\frac{L}{K} \quad \frac{\left(-K^2\right) - b\cdot R + 3\cdot\lambda^2\cdot J\cdot L}{K} \quad \frac{(-J)\cdot R - b\cdot L + 3\cdot\lambda\cdot J\cdot L}{K} \right]$$

DC Motor Speed Control

DC Motor Parameters

See http://www.engin.umich.edu/group/ctm/examples/motor/motor.html

$J := 0.01$	$J = 0.01$	kg*m^2
$b := 0.1$	$b = 0.1$	N*m*s
$K := 0.01$	$K = 0.01$	V/rad/sec
$R := 1$	$R = 1$	Ohm
$L := 0.5$	$L = 0.5$	Henrys

Find the DC gain. It is about 0.1 radians per second per volt. Therefore the motor has a maximum speed of 1 radian per second which is very slow. Obviously this is not a realistic example.

$$\lim_{s \to 0} \frac{K}{(J \cdot s + b) \cdot (L \cdot s + R) + K^2} \quad \text{float}, 2 \ \to .10$$

Calculate the PID gains. Place the three closed loop poles at -λ radians to get a critically damped response. Making λ bigger moves the closed loop poles farther to the left and makes the response faster.

$\lambda := 2 \cdot \pi \cdot 3$ $\qquad$ $\lambda = 18.849556$

$Ki := \lambda^3 \cdot J \cdot \frac{L}{K}$ $\qquad$ $Ki = 3348.677881$

$Kp := \frac{\left(-K^2\right) - b \cdot R + 3 \cdot \lambda^2 \cdot J \cdot L}{K}$ $\qquad$ $Kp = 522.948638$

$Kd := \frac{(-J) \cdot R - b \cdot L + 3 \cdot \lambda \cdot J \cdot L}{K}$ $\qquad$ $Kd = 22.274334$

Simulate modeling errors

The rnorm function generates a number with a mean of 1 and a standard deviation of σ. These numbers multiplied by the model parameters to generate simulated 'true' systems parameters.

$\sigma := 0.1$

$K := K \cdot \text{rnorm}(1, 1, \sigma)_0$ $\quad K = 0.01$

$J := J \cdot \text{rnorm}(1, 1, \sigma)_0$ $\quad J = 0.01$

$b := b \cdot \text{rnorm}(1, 1, \sigma)_0$ $\quad b = 0.1$

$R := R \cdot \text{rnorm}(1, 1, \sigma)_0$ $\quad R = 1$

$L := L \cdot \text{rnorm}(1, 1, \sigma)_0$ $\quad L = 0.5$

The DC Motor Velocity Model in State Space

$$A_c := \begin{pmatrix} -\frac{b}{J} & \frac{K}{J} \\ -\frac{K}{L} & -\frac{R}{L} \end{pmatrix} \qquad A_c = \begin{pmatrix} -10 & 1 \\ -0.02 & -2 \end{pmatrix} \qquad B_c := \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} \qquad B_c = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$I := \text{identity}(2)$

$T := .001$

Calculate arrays for use in discrete time.

$$A := I + \sum_{n=1}^{7} \frac{(A_c \cdot T)^n}{n!} \qquad B := \left[I + \sum_{n=1}^{7} \frac{(A_c \cdot T)^n}{(n+1)!} \right] \cdot B_c \cdot T$$

$$A = \begin{pmatrix} 0.99005 & 0.000994 \\ -0.00002 & 0.998002 \end{pmatrix} \qquad B = \begin{pmatrix} 9.960103 \times 10^{-7} \\ 0.001998 \end{pmatrix}$$

$$Kz_i := (Ki \quad Kp \quad Kd) \cdot \begin{pmatrix} T & 0 & 0 \\ 1 & -1 & 0 \\ \frac{1}{T} & -\frac{2}{T} & \frac{1}{T} \end{pmatrix}$$

Convert the PID gains to coefficients for the PID difference equation.

$$Kz_i = (22800.631198 \quad -45071.616402 \quad 22274.333882)$$

PID Critically Damped Control. Graph the response of the DC Motor

$x_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad u_0 := 0$ $\quad err_0 := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$PID(r,x,u,err) := \left| \begin{array}{l} err \leftarrow \begin{pmatrix} r - x_0 \\ err_0 \\ err_1 \end{pmatrix} \\ \begin{pmatrix} A \cdot x + B \cdot u \\ \max\left(\min\left(u + Kz_i \cdot err, 10\right), -10\right) \\ err \end{pmatrix} \end{array} \right.$$

Shift the error history

Update state

Calculate control output

Return the error history too

Simulate 10 seconds in increments of T seconds

$N := \frac{10}{T}$ $\quad N = 10000$ $\quad n := 0..N$

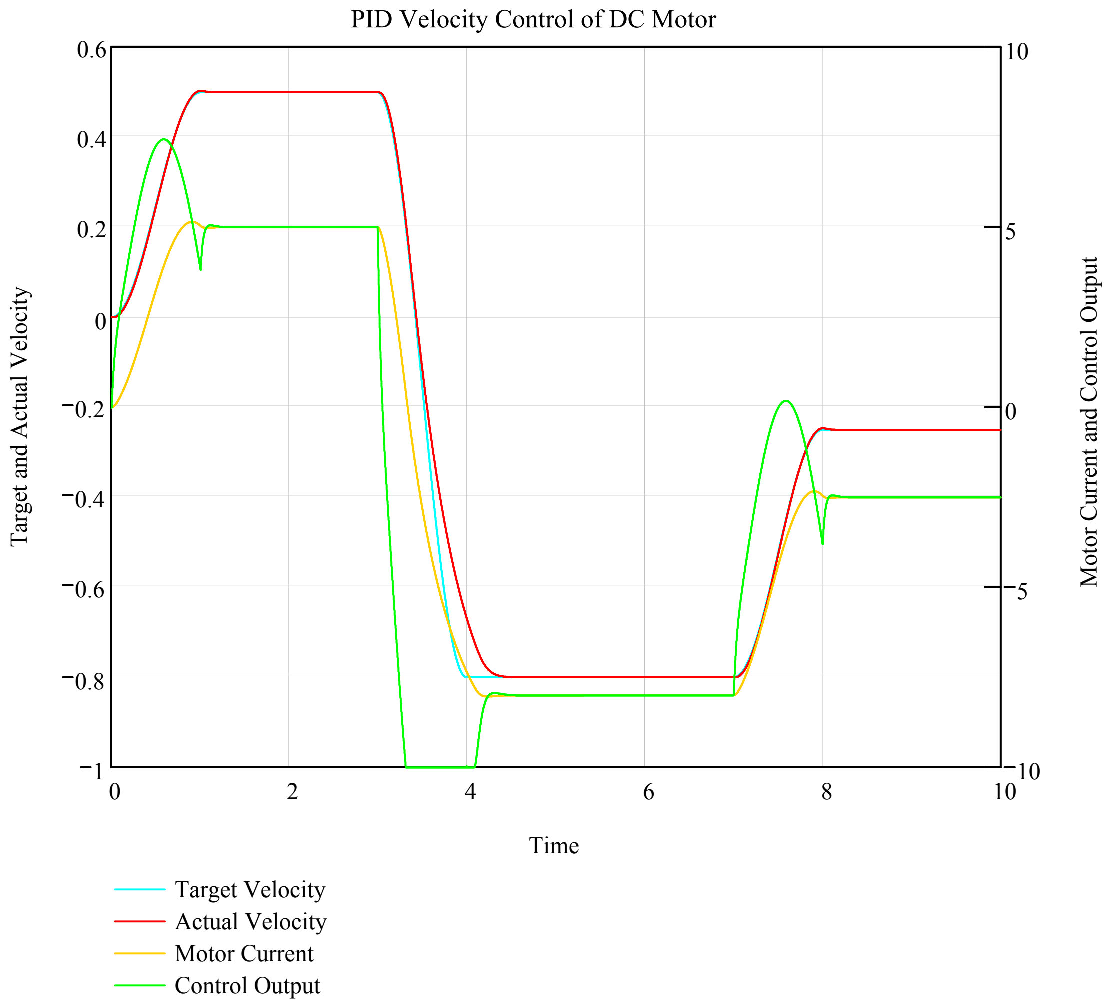
$$r_n := \left| \begin{array}{l} 0 \text{ if } n < 10 \\ \left(\frac{0.5 + 0.0}{2}\right) - \left(\frac{0.5 - 0.0}{2}\right) \cdot \cos\left[\pi \cdot \left(\frac{n - 10}{0.1 \cdot N}\right)\right] \text{ if } 10 \le n < 1010 \\ 0.5 \text{ if } n \ge 1010 \wedge n < 0.3 \cdot N \\ \left(\frac{0.5 - 0.8}{2}\right) - \left[\left(\frac{-0.8 - 0.5}{2}\right) \cdot \cos\left[\pi \cdot \left(\frac{n - 0.3 \cdot N}{0.1 \cdot N}\right)\right]\right] \text{ if } 0.3 \cdot N \le n < 0.4 \cdot N \\ -0.8 \text{ if } n \ge 0.4 \cdot N \wedge n < 0.7 \cdot N \\ \left(\frac{-0.8 - 0.25}{2}\right) - \left[\left(\frac{-0.25 - -0.8}{2}\right) \cdot \cos\left[\pi \cdot \left(\frac{n - 0.7 \cdot N}{0.1 \cdot N}\right)\right]\right] \text{ if } 0.7 \cdot N \le n < 0.8 \cdot N \\ -0.25 \text{ otherwise} \end{array} \right.$$

r(n) is the target or reference velocity.

$$\begin{pmatrix} x_{n+1} \\ u_{n+1} \\ err_{n+1} \end{pmatrix} := PID\left(r_n, x_n, u_n, err_n\right)$$

Call the PID routine for each time period.

DC Motor Speed Control



$K = 0.01$	$J = 0.01$	$R = 1$	$L = 0.5$	Motor
$\lambda = 18.849556$	$Ki = 3348.677881$	$Kp = 522.948638$	$Kd = 22.274334$	Inner Loop

The actual velocity tracks the target velocity almost perfectly execet where the control output saturates at about 3.3 seconds.   A less radical velocity change or reducing the bandwidth by reducing λ wouls avoid the output saturation.

Open Loop Bode Plot Calculations

$$T(s) := \left(\frac{Ki}{s} + Kp + Kd \cdot s\right) \cdot \frac{K}{(J \cdot s + b) \cdot (L \cdot s + R) + K^2}$$

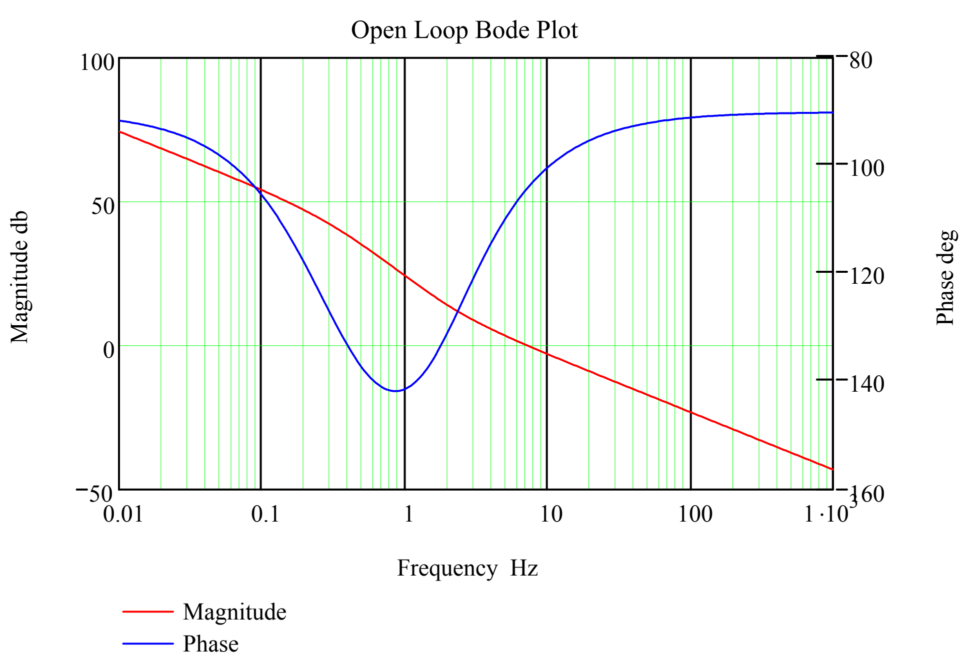
$n := 0..320$

Iterate the frequency from .01 HZ to 1000 HZ using 64 steps per decade for a total of 256 iterations.

$$Hz_n := 10^{\frac{n}{64}-2}$$

Closed loop magnitude of transfer function as a function of frequency

$$\begin{pmatrix} M_n \\ \phi_n \end{pmatrix} := \left| \begin{array}{l} r \leftarrow T\left[j \cdot (2 \cdot \pi) \cdot Hz_n\right] \\ a \leftarrow \dfrac{\arg(r)}{deg} \\ a \leftarrow a - 360 \quad \text{if } n > 0 \wedge \left|a - \phi_{n-1}\right| > 100 \\ \begin{pmatrix} 20 \cdot \log(|r|) \\ a \end{pmatrix} \end{array} \right.$$



Closed Loop Bode Plot Calculations

$$T(s) := \frac{\left(Ki + Kp\cdot s + Kd\cdot s^2\right)\cdot K}{J\cdot s^3\cdot L + J\cdot s^2\cdot R + b\cdot L\cdot s^2 + s\cdot b\cdot R + s\cdot K^2 + K\cdot Ki + K\cdot Kp\cdot s + K\cdot Kd\cdot s^2}$$

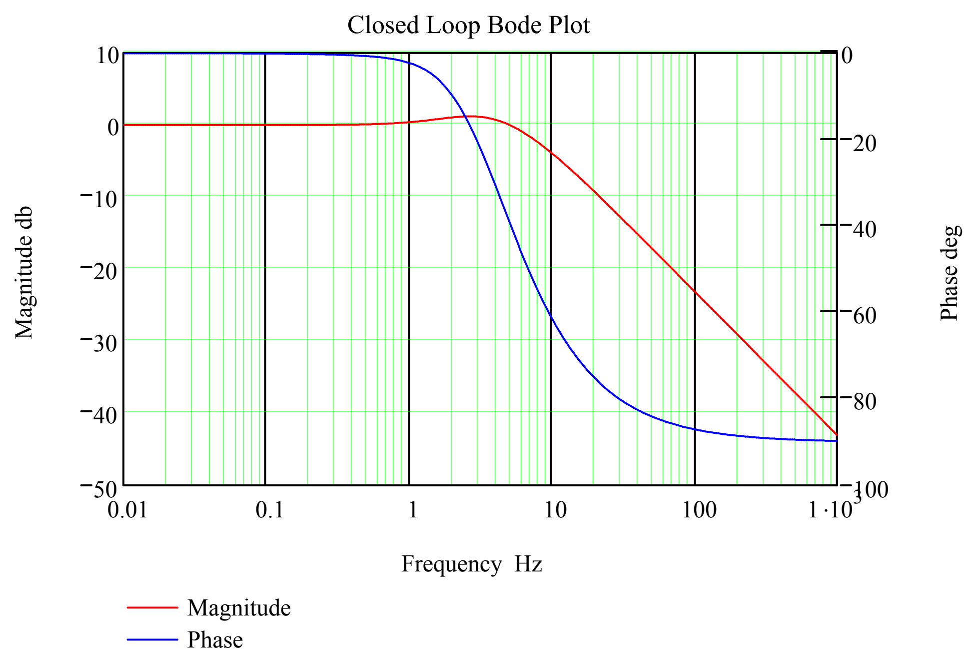
$n := 0..320$

Iterate the frequency from .01 HZ to 1000 HZ using 64 steps per decade for a total of 256 iterations.

$Hz_n := 10^{\frac{n}{64}-2}$

Closed loop magnitude of transfer function as a function of frequency

$$\begin{pmatrix} M_n \\ \phi_n \end{pmatrix} := \left| \begin{array}{l} r \leftarrow T\left[j\cdot(2\cdot\pi)\cdot Hz_n\right] \\ a \leftarrow \dfrac{\arg(r)}{deg} \\ a \leftarrow a - 360 \quad \text{if } n > 0 \wedge \left|a - \phi_{n-1}\right| > 100 \\ \begin{pmatrix} 20\cdot\log(|r|) \\ a \end{pmatrix} \end{array} \right.$$



Poles and Zeros

$$T(s) := \frac{\left(Ki + Kp \cdot s + Kd \cdot s^2\right) \cdot K}{J \cdot s^3 \cdot L + (J \cdot R + b \cdot L + K \cdot Kd) \cdot s^2 + \left(K^2 + K \cdot Kp + b \cdot R\right) \cdot s + K \cdot Ki}$$

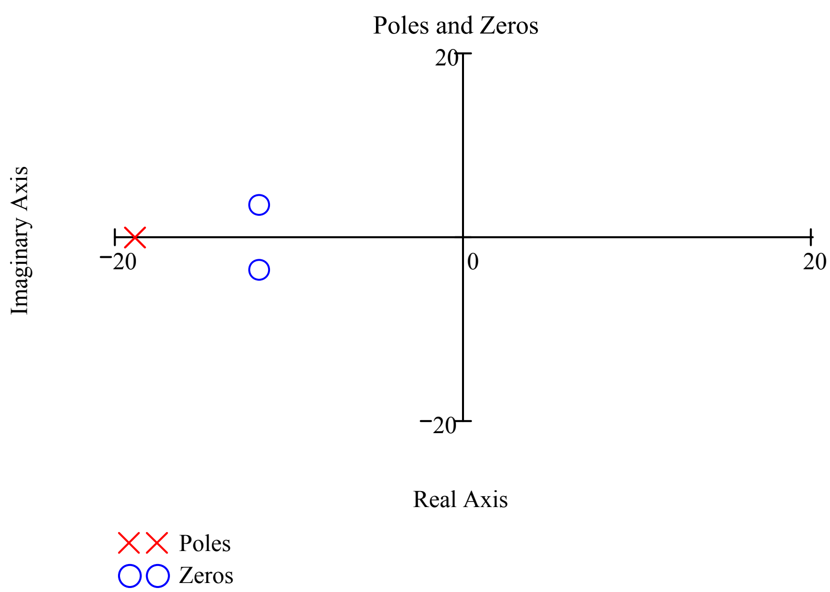
$$J \cdot s^3 \cdot L + (J \cdot R + b \cdot L + K \cdot Kd) \cdot s^2 + \left(K^2 + K \cdot Kp + b \cdot R\right) \cdot s + K \cdot Ki \ \left| \begin{array}{l} \text{coeffs}, s \\ \text{explicit} \end{array} \right. \rightarrow \begin{pmatrix} K \cdot Ki \\ K^2 + K \cdot Kp + b \cdot R \\ J \cdot R + b \cdot L + K \cdot Kd \\ J \cdot L \end{pmatrix}$$

$$poles := \text{polyroots}\left(\begin{pmatrix} K \cdot Ki \\ K^2 + K \cdot Kp + b \cdot R \\ J \cdot R + b \cdot L + K \cdot Kd \\ J \cdot L \end{pmatrix}\right) \qquad poles = \begin{pmatrix} -18.849556 \\ -18.849556 \\ -18.849555 \end{pmatrix}$$

Right where we wanted them to be!

$$\left(Ki + Kp \cdot s + Kd \cdot s^2\right) \cdot K \ \left| \begin{array}{l} \text{coeffs}, s \\ \text{explicit} \end{array} \right. \rightarrow \begin{pmatrix} K \cdot Ki \\ K \cdot Kp \\ K \cdot Kd \end{pmatrix}$$

$$zeros := \text{polyroots}\left(\begin{pmatrix} K \cdot Ki \\ K \cdot Kp \\ K \cdot Kd \end{pmatrix}\right) \qquad zeros = \begin{pmatrix} -11.738817 - 3.540924i \\ -11.738817 + 3.540924i \end{pmatrix}$$



Add Outer Position Loop

$\lambda := \lambda \qquad Ki_i := Ki \qquad Kp_i := Kp \qquad Kd_i := Kd$

Use i subscripts for inner loop parameters and i subscripts for outer loop parameters.

$$T_o = \frac{\left(\frac{Ki_o}{s} + Kp_o + Kd_o\cdot s\right)\cdot\frac{\lambda^3}{s\cdot(s+\lambda)^3}}{1 + \left(\frac{Ki_o}{s} + Kp_o + Kd_o\cdot s\right)\cdot\frac{\lambda^3}{s\cdot(s+\lambda)^3}}$$

$$T_o = \frac{\left(Ki_o + Kp_o\cdot s + Kd_o\cdot s^2\right)\cdot\lambda^3}{s^5 + 3\cdot s^4\cdot\lambda + 3\cdot s^3\cdot\lambda^2 + \left(\lambda^3\cdot Kd_o + \lambda^3\right)\cdot s^2 + \lambda^3\cdot Kp_o\cdot s + \lambda^3\cdot Ki_o}$$

$$DiffCE := s^5 + 3\cdot s^4\cdot\lambda + 3\cdot s^3\cdot\lambda^2 + \left(\lambda^3\cdot Kd_o + \lambda^3\right)\cdot s^2 + \lambda^3\cdot Kp_o\cdot s + \lambda^3\cdot Ki_o - (s+\mu)^3\cdot(s+\alpha)\cdot(s+\beta)$$

$$DiffCE \left|\begin{array}{l} \text{coeffs}, s \\ \text{solve}, \begin{pmatrix} Ki_o \\ Kp_o \\ Kd_o \\ \alpha \\ \beta \end{pmatrix} \end{array}\right. \rightarrow \left[\begin{array}{lll} 3\cdot\mu^3\cdot\frac{\lambda^2 - 3\cdot\mu\cdot\lambda + 2\cdot\mu^2}{\lambda^3} & 3\cdot\mu^2\cdot\frac{(-8)\cdot\mu\cdot\lambda + 5\cdot\mu^2 + 3\cdot\lambda^2}{\lambda^3} & \frac{-(\lambda^3 + 18\cdot\mu^2\cdot\lambda - 1(}{\lambda^3} \\ 3\cdot\mu^3\cdot\frac{\lambda^2 - 3\cdot\mu\cdot\lambda + 2\cdot\mu^2}{\lambda^3} & 3\cdot\mu^2\cdot\frac{(-8)\cdot\mu\cdot\lambda + 5\cdot\mu^2 + 3\cdot\lambda^2}{\lambda^3} & \frac{-(\lambda^3 + 18\cdot\mu^2\cdot\lambda - 1(}{\lambda^3} \end{array}\right.$$

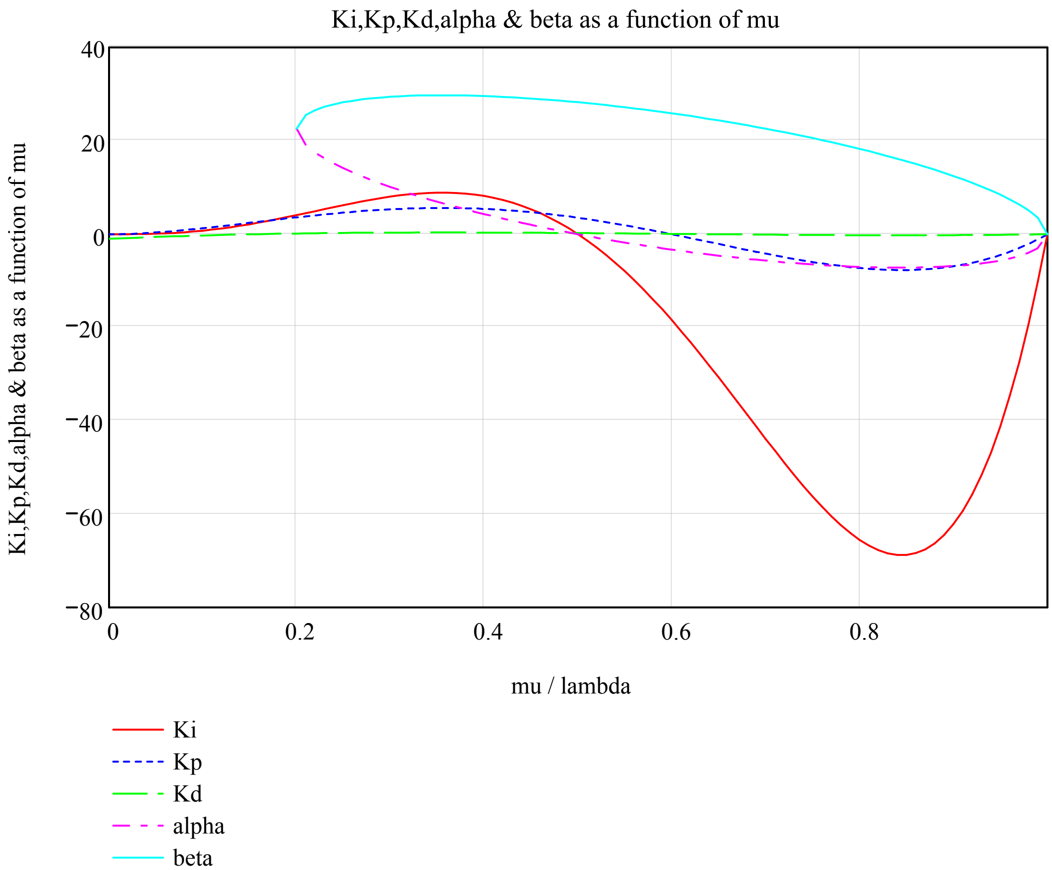
$$\begin{pmatrix} Ki_o \\ Kp_o \\ Kd_o \\ \alpha \\ \beta \end{pmatrix} = \begin{bmatrix} 3\cdot\mu^3\cdot\frac{\lambda^2 - 3\cdot\mu\cdot\lambda + 2\cdot\mu^2}{\lambda^3} \\ 3\cdot\mu^2\cdot\frac{5\cdot\mu^2 - 8\cdot\mu\cdot\lambda + 3\cdot\lambda^2}{\lambda^3} \\ \frac{10\cdot\mu^3 - 18\cdot\mu^2\cdot\lambda - \lambda^3 + 9\cdot\mu\cdot\lambda^2}{\lambda^3} \\ \frac{3}{2}\cdot\lambda - \frac{3}{2}\cdot\mu - \frac{1}{2}\cdot\left[(-3)\cdot\lambda^2 + 18\cdot\mu\cdot\lambda - 15\cdot\mu^2\right]^{\frac{1}{2}} \\ \frac{3}{2}\cdot\lambda - \frac{3}{2}\cdot\mu + \frac{1}{2}\cdot\left[(-3)\cdot\lambda^2 + 18\cdot\mu\cdot\lambda - 15\cdot\mu^2\right]^{\frac{1}{2}} \end{bmatrix}$$

The symbolic equations for the outer loop gains as a function of the inner loop poles and the outer loop poles. If the relationship between the outer loop poles, μ, and the inner loop poles, λ, is not right the outer loop gains will be negative or the unplaced poles, α and β, will be in the right hand plane.

DC Motor Speed Control

Graphs the outer loop gains and α and β as a function of the ratio of the outer loop pole location to the inner loop pole locations. This is backwards from a root locus plot.

$\mu := 0, 0.01\lambda .. 1\lambda$



μ is only valid where Ki, Kp, Kd, α and β are positive and this is only a small fractional range of λ.
Kd is negative if μ/λ is too low and α and β are negative if μ/λ is greater than 1.
Ki, Kp and Kd are maximum at 0.355λ:

$\mu := \mu$

Find μ for max Ki

$$\frac{d}{d\mu}\left(3\cdot\mu^3\cdot\frac{\lambda^2-3\cdot\mu\cdot\lambda+2\cdot\mu^2}{\lambda^3}\right)\text{ solve},\mu \rightarrow \begin{bmatrix} 0 \\ 0 \\ \left(\frac{3}{5}+\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda \\ \left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ \left(\frac{3}{5}+\frac{1}{10}\cdot 6^{\frac{1}{2}}\right) \\ \left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right) \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.844949 \\ 0.355051 \end{pmatrix}$$

Find μ for max Kp

$$\frac{d}{d\mu}\left(3\cdot\mu^2\cdot\frac{5\cdot\mu^2-8\cdot\mu\cdot\lambda+3\cdot\lambda^2}{\lambda^3}\right)\text{ solve},\mu \rightarrow \begin{bmatrix} 0 \\ \left(\frac{3}{5}+\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda \\ \left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda \end{bmatrix} \quad \begin{bmatrix} 0 \\ \left(\frac{3}{5}+\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda \\ \left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda \end{bmatrix} = \begin{pmatrix} 0 \\ 15.926913 \\ 6.692554 \end{pmatrix}$$

Find μ for max Kd

$$\frac{d}{d\mu}\left(3\cdot\frac{10\cdot\mu^3-18\cdot\mu^2\cdot\lambda-\lambda^3+9\cdot\mu\cdot\lambda^2}{\lambda^3}\right)\text{ solve},\mu \rightarrow \begin{bmatrix} \left(\frac{3}{5}+\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda \\ \left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda \end{bmatrix}$$

The maximum outer loop gains will all happen at the same ratio of μ to λ.

$\mu := \left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda$ or about 0.355*λ

$$\mu := \left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda \qquad \mu = 6.692554$$

Substitute the optimal value of μ in terms of λ

Now all tuning can be done in terms of the inner loop poles position using λ alone.

$$\begin{pmatrix} Ki_o \\ Kp_o \\ Kd_o \\ \alpha \\ \beta \end{pmatrix} = \begin{bmatrix} 3\cdot\mu^3\cdot\dfrac{\lambda^2 - 3\cdot\mu\cdot\lambda + 2\cdot\mu^2}{\lambda^3} \\ 3\cdot\mu^2\cdot\dfrac{5\cdot\mu^2 - 8\cdot\mu\cdot\lambda + 3\cdot\lambda^2}{\lambda^3} \\ \dfrac{10\cdot\mu^3 - 18\cdot\mu^2\cdot\lambda - \lambda^3 + 9\cdot\mu\cdot\lambda^2}{\lambda^3} \\ \dfrac{3}{2}\cdot\lambda - \dfrac{3}{2}\cdot\mu - \dfrac{1}{2}\cdot\left[(-3)\cdot\lambda^2 + 18\cdot\mu\cdot\lambda - 15\cdot\mu^2\right]^{\frac{1}{2}} \\ \dfrac{3}{2}\cdot\lambda - \dfrac{3}{2}\cdot\mu + \dfrac{1}{2}\cdot\left[(-3)\cdot\lambda^2 + 18\cdot\mu\cdot\lambda - 15\cdot\mu^2\right]^{\frac{1}{2}} \end{bmatrix} \left| \begin{array}{l} \text{substitute}, \mu = \left(\dfrac{3}{5} - \dfrac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda \\ \text{expand} \\ \text{simplfy} \\ \text{collect}, \lambda \end{array} \right. \rightarrow \begin{pmatrix} Ki_o \\ Kp_o \\ Kd_o \\ \alpha \\ \beta \end{pmatrix} =$$

The formulas for outer loop gains and unplaced poles in terms of λ alone

$$\begin{pmatrix} Ki_o \\ Kp_o \\ Kd_o \\ \alpha \\ \beta \end{pmatrix} = \begin{bmatrix} 3\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^3\cdot\left[\frac{-4}{5}+\frac{3}{10}\cdot 6^{\frac{1}{2}}+2\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^2\right]\cdot\lambda^2 \\ 3\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^2\cdot\left[5\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^2-\frac{9}{5}+\frac{4}{5}\cdot 6^{\frac{1}{2}}\right]\cdot\lambda \\ 10\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^3-18\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^2+\frac{22}{5}-\frac{9}{10}\cdot 6^{\frac{1}{2}} \\ \left(\frac{3}{5}+\frac{3}{20}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda-\frac{1}{2}\cdot\left[(-3)\cdot\lambda^2+18\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda^2-15\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^2\cdot\lambda^2\right]^{\frac{1}{2}} \\ \left(\frac{3}{5}+\frac{3}{20}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda+\frac{1}{2}\cdot\left[(-3)\cdot\lambda^2+18\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda^2-15\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^2\cdot\lambda^2\right]^{\frac{1}{2}} \end{bmatrix}$$

Calculate numerical values for the outer loop gains

$\lambda = 18.849556$ Three Inner loop poles at $-\lambda$

$\mu := 0.45 \cdot \lambda$ Enable to override optimum outer loop gains.

$\mu = 6.692554$ Three Outer loop poles at $-\mu$

$$\begin{pmatrix} Ki_o \\ Kp_o \\ Kd_o \\ \alpha \\ \beta \end{pmatrix} := \begin{bmatrix} 3 \cdot \mu^3 \cdot \dfrac{\lambda^2 - 3 \cdot \mu \cdot \lambda + 2 \cdot \mu^2}{\lambda^3} \\ 3 \cdot \mu^2 \cdot \dfrac{5 \cdot \mu^2 - 8 \cdot \mu \cdot \lambda + 3 \cdot \lambda^2}{\lambda^3} \\ \dfrac{10 \cdot \mu^3 - 18 \cdot \mu^2 \cdot \lambda - \lambda^3 + 9 \cdot \mu \cdot \lambda^2}{\lambda^3} \\ \dfrac{3}{2} \cdot \lambda - \dfrac{3}{2} \cdot \mu - \dfrac{1}{2} \cdot \left[(-3) \cdot \lambda^2 + 18 \cdot \mu \cdot \lambda - 15 \cdot \mu^2\right]^{\frac{1}{2}} \\ \dfrac{3}{2} \cdot \lambda - \dfrac{3}{2} \cdot \mu + \dfrac{1}{2} \cdot \left[(-3) \cdot \lambda^2 + 18 \cdot \mu \cdot \lambda - 15 \cdot \mu^2\right]^{\frac{1}{2}} \end{bmatrix}$$

$$\begin{pmatrix} Ki_o \\ Kp_o \\ Kd_o \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 8.92003 \\ 5.630862 \\ 0.373939 \\ 6.692554 \\ 29.778451 \end{pmatrix}$$

α is the same as μ so there are four outer loop poles at μ and one at β. The pole at β is always farther to the left so it doesn't affect the response much.

The DC Motor Velocity Model in State Space

$$A_c := \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{K}{J} \\ 0 & -\frac{K}{L} & -\frac{R}{L} \end{pmatrix} \qquad A_c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -10 & 1 \\ 0 & -0.02 & -2 \end{pmatrix} \qquad B_c := \begin{pmatrix} 0 \\ 0 \\ \frac{1}{L} \end{pmatrix} \qquad B_c = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$I := \text{identity}(3)$$

Calculate arrays for use in discrete time.

$$A := I + \sum_{n=1}^{7} \frac{(A_c \cdot T)^n}{n!} \qquad B := \left[I + \sum_{n=1}^{7} \frac{(A_c \cdot T)^n}{(n+1)!} \right] \cdot B_c \cdot T$$

$$A = \begin{pmatrix} 1 & 0.000995 & 4.980052 \times 10^{-7} \\ 0 & 0.99005 & 0.000994 \\ 0 & -0.00002 & 0.998002 \end{pmatrix} \qquad B = \begin{pmatrix} 3.323354 \times 10^{-10} \\ 9.960103 \times 10^{-7} \\ 0.001998 \end{pmatrix}$$

Cascade Control

The Cascaded loop controller and system simulator.
An outer loop closed loop generates a reference velocity. The inner loop tries to follow the velocity generated by the outer loop.
Note the outer and inner loop is really using I-PD control where only the integrator term is in the forward path. This makes the inner loop less susceptible to the noise in the set point from the outer loop.
The second line computes the control output using the error between velocity generated from the outer loop and the feed back or estimated velocity. The control output is limited to -10 to +10 volts

$$N := \frac{10}{T} \qquad n := 0 .. N$$

Simulate 10 seconds

$$r_n := \begin{array}{|l} x_0 \ \text{ if } \ n < 0.01 \cdot N \\ \begin{pmatrix} 0.2 \\ 0 \end{pmatrix} \ \text{ if } \ 0.01 \cdot N \le n \wedge n < 0.2 \cdot N \\ \begin{pmatrix} 0.1 \\ 0 \end{pmatrix} \ \text{ if } \ 0.2 \cdot N \le n \wedge n < 0.4 \cdot N \\ \begin{pmatrix} 0.05 \\ 0 \end{pmatrix} \ \text{ if } \ 0.4 \cdot N \le n \wedge n < 0.65 \cdot N \\ \begin{pmatrix} 0.35 \\ 0 \end{pmatrix} \ \text{ otherwise} \end{array}$$

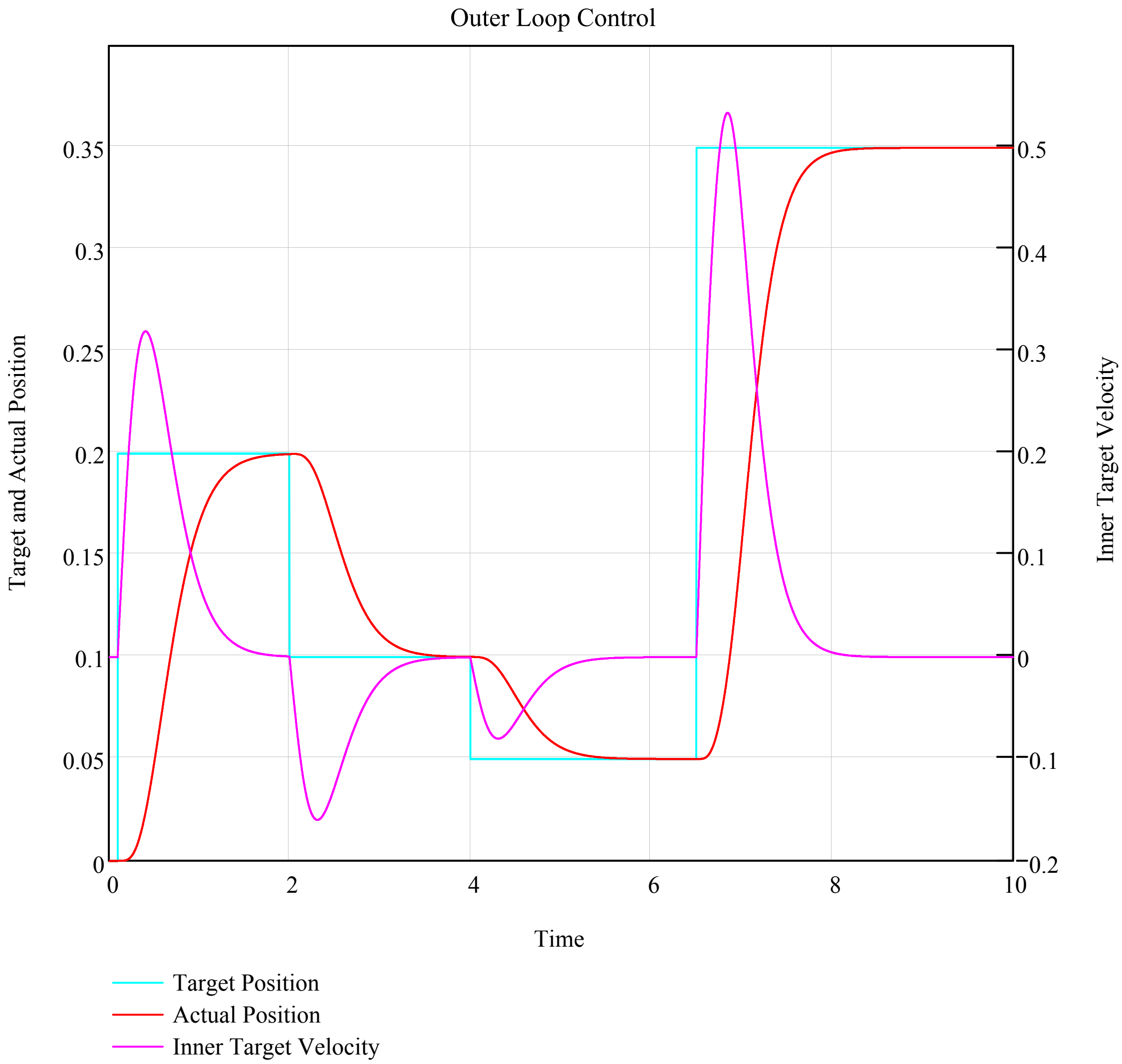
Calculate the target positions and velocities. In this case the target velocities are 0 since that target positions are changing only in step jumps.

$$
\text{Cascade}(r) := \left|
\begin{array}{l}
x_0 \leftarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
v_0 \leftarrow 0 \\
v_i \leftarrow 0 \\
u_0 \leftarrow 0 \\
u_i \leftarrow 0 \\
\text{for } n \in 0..N-1 \\
\quad \left|
\begin{array}{l}
x_{n+1} \leftarrow A \cdot x_n + B \cdot u_n \\
v_{pd} \leftarrow -\left[Kp_o \cdot \left(x_{n+1}\right)_0 + \frac{Kd_o}{T} \cdot \left[\left(x_{n+1}\right)_0 - \left(x_n\right)_0\right]\right] \\
v_i \leftarrow \max\left[\min\left[v_i + Ki_o \cdot T \cdot \left[\left(r_{n+1}\right)_0 - \left(x_{n+1}\right)_0\right], 1.0 - v_{pd}\right], -1.0 - v_{pd}\right] \\
v_{n+1} \leftarrow v_i + v_{pd} \\
u_{pd} \leftarrow -\left[Kp_i \cdot \left(x_{n+1}\right)_1 + \frac{Kd_i}{T} \cdot \left[\left(x_{n+1}\right)_1 - \left(x_n\right)_1\right]\right] \\
u_i \leftarrow \max\left[\min\left[u_i + Ki_i \cdot T \cdot \left[v_{n+1} - \left(x_{n+1}\right)_1\right], 10 - u_{pd}\right], -10 - u_{pd}\right] \\
u_{n+1} \leftarrow u_i + u_{pd}
\end{array}
\right. \\
\begin{pmatrix} x \\ u \\ v \end{pmatrix}
\end{array}
\right.
$$

$$
\begin{pmatrix} x \\ u \\ v \end{pmatrix} := \text{Cascade}(r)
$$

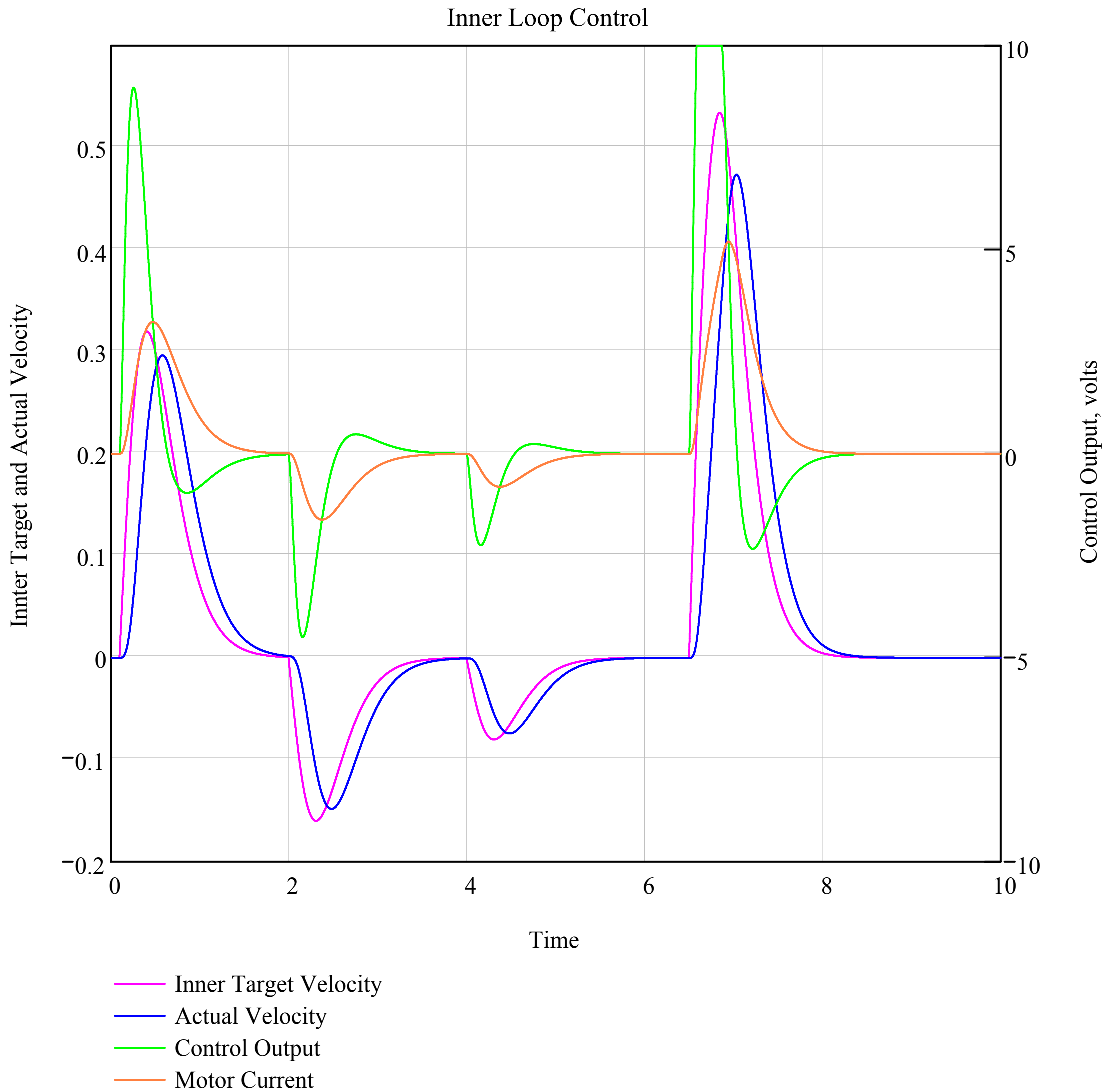
Compute the system response and control output

Verify By Simulating and Graphing the Results.



$K = 0.01$	$J = 0.01$	$R = 1$	$L = 0.5$	Motor
$\lambda = 18.849556$	$Ki_i = 3348.677881$	$Kp_i = 522.948638$	$Kd_i = 22.274334$	Inner Loop
$\mu = 6.692554$	$Ki_o = 8.92003$	$Kp_o = 5.630862$	$Kd_o = 0.373939$	Outer Loop

DC Motor Speed Control



$K = 0.01$	$J = 0.01$	$R = 1$	$L = 0.5$	Motor
$\lambda = 18.849556$	$Ki_i = 3348.677881$	$Kp_i = 522.948638$	$Kd_i = 22.274334$	Inner Loop
$\mu = 6.692554$	$Ki_o = 8.92003$	$Kp_o = 5.630862$	$Kd_o = 0.373939$	Outer Loop

DC Motor Speed Control

Cascade Control of a Type 1 Single Pole System Following a Sine Wave

$Amp := 0.5 \qquad Hz := 0.25$

Amplitude and frequency of target generator.

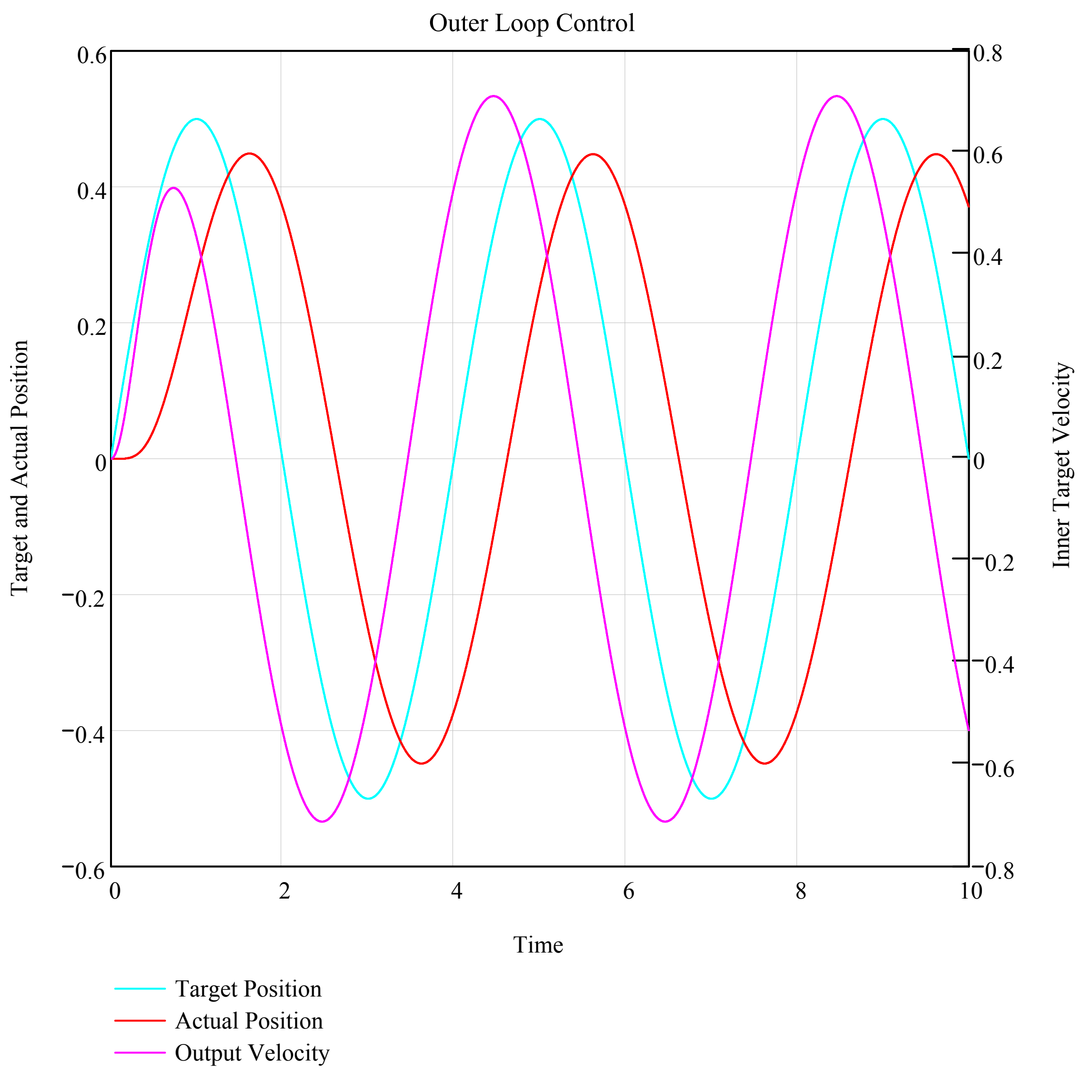
Initial state of position and velocity

$$r_n := \begin{pmatrix} Amp \cdot \sin(2 \cdot \pi \cdot Hz \cdot n \cdot T) \\ 2 \cdot \pi \cdot Hz \cdot Amp \cdot \cos(2 \cdot \pi \cdot Hz \cdot n \cdot T) \end{pmatrix}$$

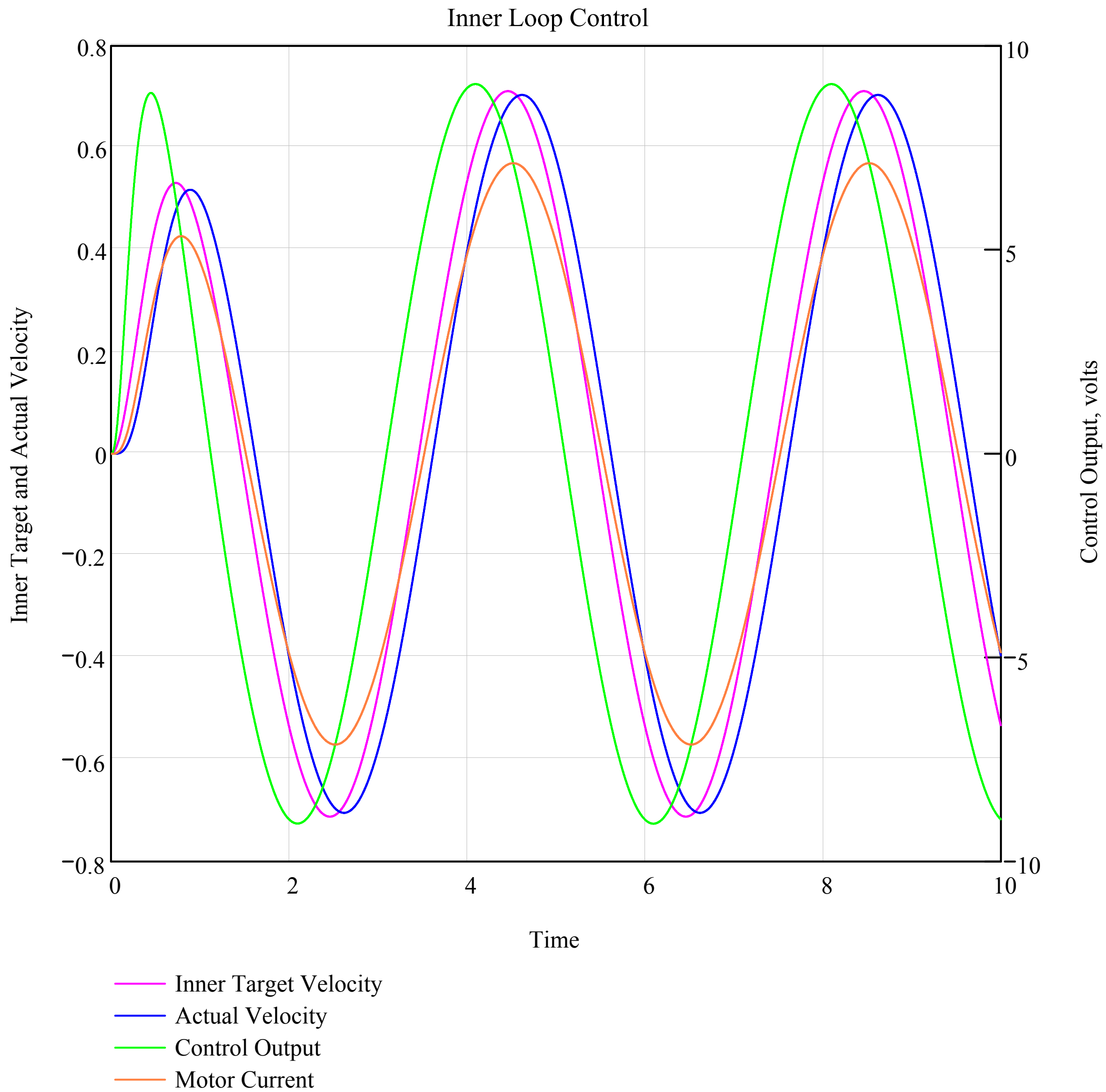
Sinusoidal motion generator. Generates target position, velocity and acceleration.

$$\begin{pmatrix} x \\ u \\ v \end{pmatrix} := Cascade(r)$$

Compute the system response and control output



DC Motor Speed Control



$K = 0.01$	$J = 0.01$	$R = 1$	$L = 0.5$	Actuator
$\lambda = 18.849556$	$Ki_i = 3348.677881$	$Kp_i = 522.948638$	$Kd_i = 22.274334$	Inner Loop
$\mu = 6.692554$	$Ki_o = 8.92003$	$Kp_o = 5.630862$	$Kd_o = 0.373939$	Outer Loop

Calculate Bandwidths

Calculate Inner Loop Bandwidth

$$f(\omega) := \left| \frac{\left(\frac{Ki_i}{j\cdot\omega}\right)\cdot\frac{K}{[J\cdot(j\cdot\omega)+b]\cdot[L\cdot(j\cdot\omega)+R]+K^2}}{1+\left[\frac{Ki_i}{j\cdot\omega}+Kp_i+Kd_i\cdot(j\cdot\omega)\right]\cdot\frac{K}{[J\cdot(j\cdot\omega)+b]\cdot[L\cdot(j\cdot\omega)+R]+K^2}} \right|$$

$\omega_i := 0.7\cdot\lambda$ $\quad\omega_i = 13.194689$ The initial guess for the inner loop bandwidth

$\omega_i := \mathrm{root}\left(f(\omega_i) - \frac{1}{\sqrt{2}}, \omega_i\right)$ $\quad\omega_i = 9.609966$ Inner loop bandwidth in radians per second

$\frac{\omega_i}{2\cdot\pi} = 1.529474$ Inner loop bandwidth in Hz

Calculate Outer Loop Bandwidth

$$f(\omega) := \left| \frac{\left(\frac{Ki_o}{j\cdot\omega}\right)\cdot\frac{1}{j\cdot\omega}\cdot\frac{\left(\frac{Ki_i}{j\cdot\omega}\right)\cdot\frac{K}{[J\cdot(j\cdot\omega)+b]\cdot[L\cdot(j\cdot\omega)+R]+K^2}}{1+\left[\frac{Ki_i}{j\cdot\omega}+Kp_i+Kd_i\cdot(j\cdot\omega)\right]\cdot\frac{K}{[J\cdot(j\cdot\omega)+b]\cdot[L\cdot(j\cdot\omega)+R]+K^2}}}{1+\left[\frac{Ki_o}{j\cdot\omega}+Kp_o+Kd_o\cdot(j\cdot\omega)\right]\cdot\frac{1}{j\cdot\omega}\cdot\frac{\left(\frac{Ki_i}{j\cdot\omega}\right)\cdot\frac{K}{[J\cdot(j\cdot\omega)+b]\cdot[L\cdot(j\cdot\omega)+R]+K^2}}{1+\left[\frac{Ki_i}{j\cdot\omega}+Kp_i+Kd_i\cdot(j\cdot\omega)\right]\cdot\frac{K}{[J\cdot(j\cdot\omega)+b]\cdot[L\cdot(j\cdot\omega)+R]+}}}$$

$\omega_o := 0.2\lambda$ $\quad\omega_o = 3.769911$ Initial guess for the outer loop band width

$\omega_o := \mathrm{root}\left(f(\omega_o) - \frac{1}{\sqrt{2}}, \omega_o\right)$ $\quad\omega_o = 2.889634$ Outer loop bandwidth in radians per second

$\frac{\omega_o}{2\cdot\pi} = 0.459899$ outer loop bandwidth in Hz

$\frac{\omega_i}{\omega_o} = 3.325669$ Ratio of the inner loop to outer loop bandwidth. This ratio is fixed

Inner Loop Bode Plot Calculations

$$T(s) := \frac{\left(\frac{Ki_i}{s}\right)\cdot\frac{K}{(J\cdot s + b)\cdot(L\cdot s + R) + K^2}}{1 + \left(\frac{Ki_i}{s} + Kp_i + Kd_i\cdot s\right)\cdot\frac{K}{(J\cdot s + b)\cdot(L\cdot s + R) + K^2}}$$

Kp_i and Kd_i are in the feed back path only. This makes the inner loop less susceptible to the noisy target velocities generated by the outer loop.

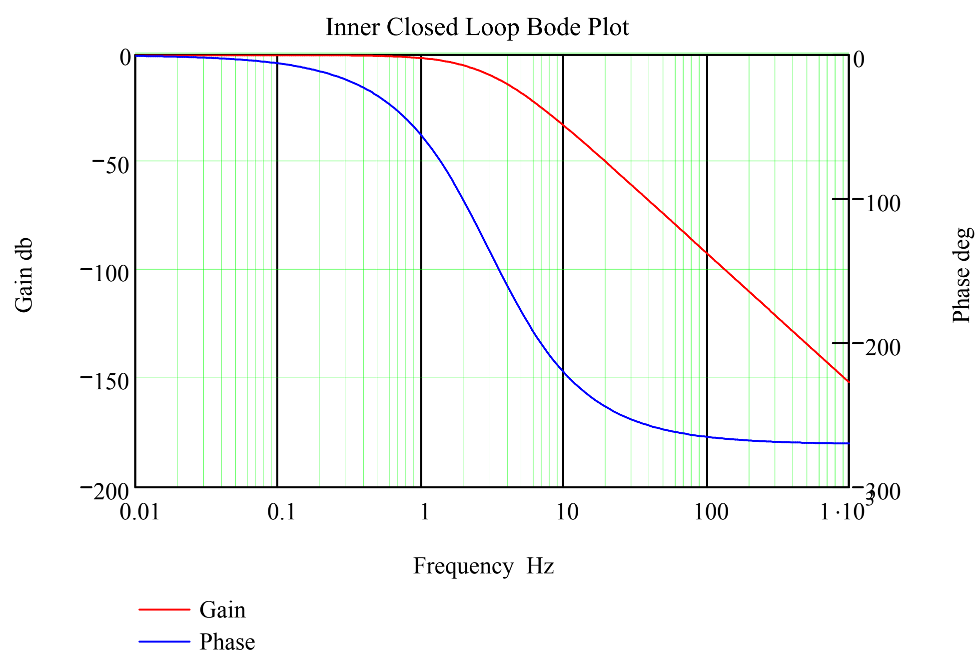
$$n := 0..320$$

$$hz_n := 10^{\frac{n}{64}-2}$$

Iterate the frequency from .01 HZ to 1000 HZ using 64 steps per decade for a total of 320 iterations.

$$\begin{pmatrix} M_n \\ \phi_n \end{pmatrix} := \left| \begin{array}{l} r \leftarrow T\left[j\cdot(2\cdot\pi)\cdot hz_n\right] \\ a \leftarrow \frac{\arg(r)}{\deg} \\ a \leftarrow a - 360 \quad \text{if } n > 0 \wedge \left|a - \phi_{n-1}\right| > 100 \\ \begin{pmatrix} 20\cdot\log(|r|) \\ a \end{pmatrix} \end{array} \right.$$

Closed loop magnitude and phase as a function of frequency



Outer Loop Bode Plot Calculations

$$T(s) := \frac{\left(\frac{Ki_o}{s}\right)\cdot\frac{\lambda^3}{s\cdot(s+\lambda)^3}}{1+\left(\frac{Ki_o}{s}+Kp_o+Kd_o\cdot s\right)\cdot\frac{\lambda^3}{s\cdot(s+\lambda)^3}}$$

The outer loop gains are in the forward and feedback path. This help makes the outer loop faster than the inner loop.

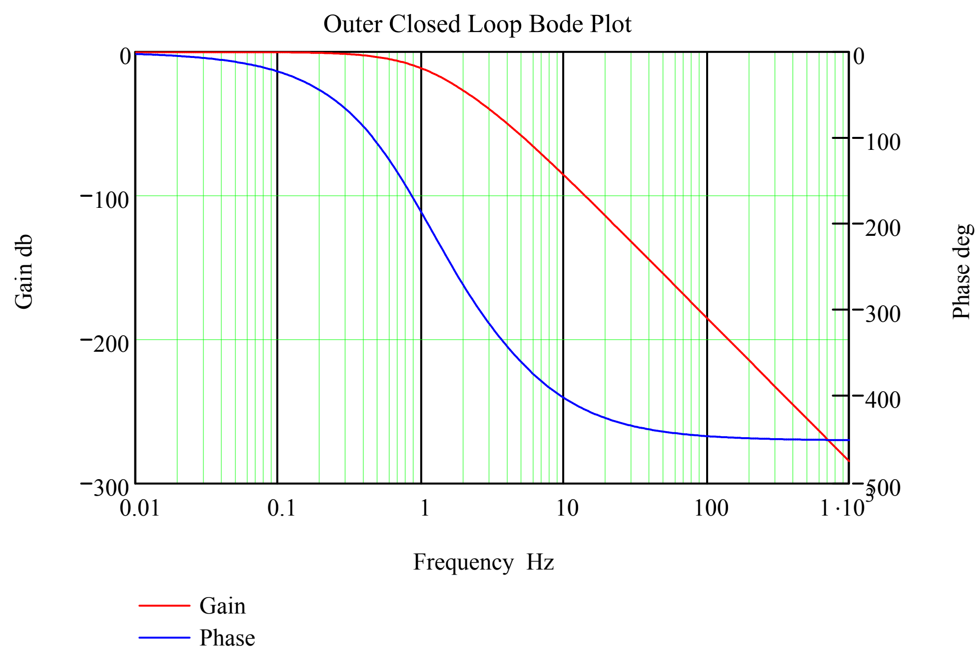
$$n := 0..320$$

Iterate the frequency from .01 HZ to 1000 HZ using 64 steps per decade for a total of 320 iterations.

$$hz_n := 10^{\frac{n}{64}-2}$$

$$\begin{pmatrix} M_n \\ \phi_n \end{pmatrix} := \left| \begin{array}{l} r \leftarrow T\left[j\cdot(2\cdot\pi)\cdot hz_n\right] \\ a \leftarrow \frac{\arg(r)}{deg} \\ a \leftarrow a-360 \quad \text{if } n>0 \wedge \left|a-\phi_{n-1}\right|>100 \\ \begin{pmatrix} 20\cdot\log(|r|) \\ a \end{pmatrix} \end{array} \right.$$

Closed loop magnitude and phase as a function of frequency



Graph the Outer Loop Poles and Zeros

$\lambda = 18.849556 \quad Ki_i = 3348.677881 \quad Kp_i = 522.948638 \quad Kd_i = 22.274334$

$\mu = 6.692554 \quad Ki_o = 8.92003 \quad Kp_o = 5.630862 \quad Kd_o = 0.373939$

$$T_i(s) := \frac{\left(Ki + Kp\cdot s + Kd\cdot s^2\right)\cdot K}{J\cdot s^3\cdot L + J\cdot s^2\cdot R + b\cdot L\cdot s^2 + s\cdot b\cdot R + s\cdot K^2 + K\cdot Ki + K\cdot Kp\cdot s + K\cdot Kd\cdot s^2} \;\Bigg|\begin{array}{l}\text{simplify}\\ \text{explicit}\end{array} \rightarrow \left(Ki + Kp\cdot s + K\right.$$

$$T_o(s) := \frac{\left(\frac{Ki_o}{s}\right)\cdot\frac{1}{s}\cdot\frac{\left(Ki_i + Kp_i\cdot s + Kd_i\cdot s^2\right)\cdot K}{J\cdot s^3\cdot L + \left(J\cdot R + b\cdot L + K\cdot Kd_i\right)\cdot s^2 + \left(K^2 + K\cdot Kp_i + b\cdot R\right)\cdot s + K\cdot Ki_i}}{1 + \left(\frac{Ki_o}{s} + Kp_o + Kd_o\cdot s\right)\cdot\frac{1}{s}\cdot\frac{\left(Ki_i + Kp_i\cdot s + Kd_i\cdot s^2\right)\cdot K}{J\cdot s^3\cdot L + \left(J\cdot R + b\cdot L + K\cdot Kd_i\right)\cdot s^2 + \left(K^2 + K\cdot Kp_i + b\cdot R\right)\cdot s + K\cdot Ki}}$$

$$A_i := \text{denom}\left(T_i(s)\right) \;\Bigg|\begin{array}{l}\text{coeffs}, s\\ \text{float}, 7\end{array} \rightarrow \begin{pmatrix} 33.48679 \\ 5.329588 \\ .2827434 \\ .5e\text{-}2 \end{pmatrix}$$

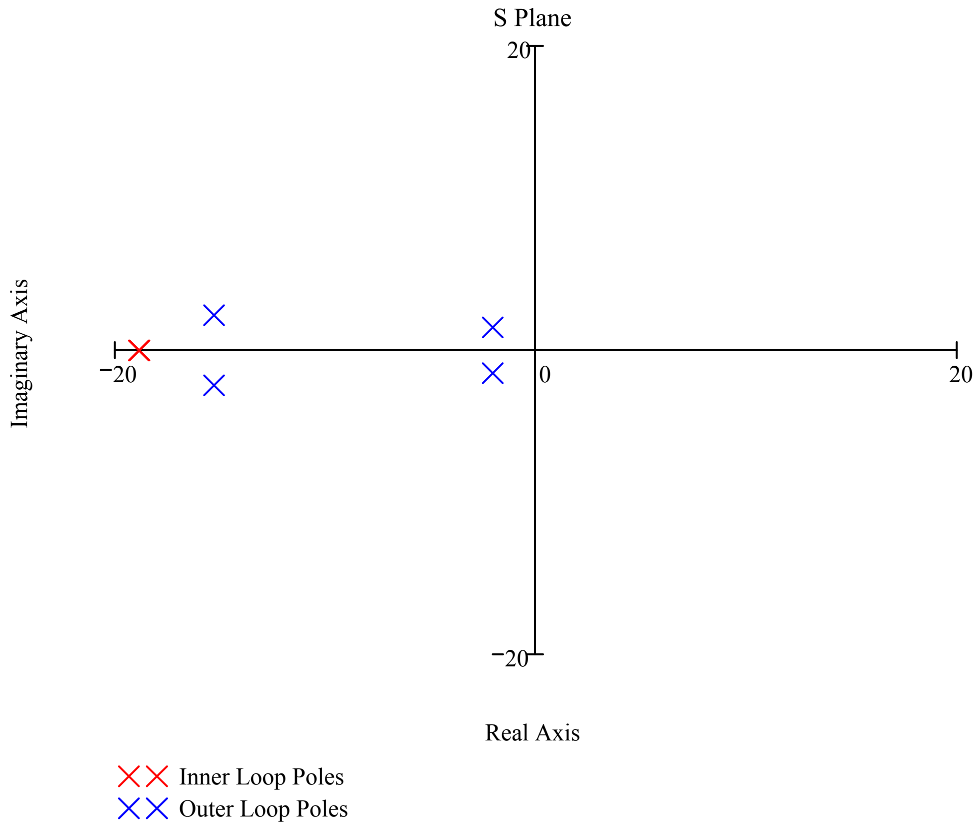
$$A_o := \text{denom}\left(T_o(s)\right) \;\Bigg|\begin{array}{l}\text{coeffs}, s\\ \text{float}, 7\end{array} \rightarrow \begin{bmatrix} (-.840693)\cdot\lambda^5 \\ (-.1312876)\cdot\lambda^5 - 10.00339\cdot\lambda^4 \\ (-1.562186)\cdot\lambda^4 - .559203e\text{-}2\cdot\lambda^5 - 46.00881\cdot\lambda^3 \\ (-7.285097)\cdot\lambda^3 - .6653937e\text{-}1\cdot\lambda^4 \\ (-.3660357)\cdot\lambda^3 \\ (-.5e\text{-}2)\cdot\lambda^3 \end{bmatrix}$$

$$B_o := \text{numer}\left(T_o(s)\right) \;\Bigg|\begin{array}{l}\text{coeffs}, s\\ \text{float}, 6\end{array} \rightarrow \begin{bmatrix} (-.84070)\cdot\lambda^5 \\ (-.131287)\cdot\lambda^5 \\ (-.559199e\text{-}2)\cdot\lambda^5 \end{bmatrix}$$

$\text{poles}_i := \text{polyroots}\left(A_i\right) \qquad \text{poles}_o := \text{polyroots}\left(A_o\right)$

$$\text{poles}_i = \begin{pmatrix} -18.862727 \\ -18.845535 \\ -18.840418 \end{pmatrix} \qquad \text{poles}_o = \begin{pmatrix} -38.497281 \\ -15.302026 - 2.312133i \\ -15.302026 + 2.312133i \\ -2.052903 + 1.505003i \\ -2.052903 - 1.505003i \end{pmatrix}$$

DC Motor Speed Control



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λ = 18.849556	Ki_i = 3348.677881	Kp_i = 522.948638	Kd_i = 22.274334	Inner Loop
μ = 6.692554	Ki_o = 8.92003	Kp_o = 5.630862	Kd_o = 0.373939	Outer Loop

$$\frac{\cdot\mu^3 - 9\cdot\mu\cdot\lambda^2)}{} \quad \frac{-3}{2}\cdot\mu + \frac{3}{2}\cdot\lambda - \frac{1}{2}\cdot\left[(-15)\cdot\mu^2 + 18\cdot\mu\cdot\lambda - 3\cdot\lambda^2\right]^{\frac{1}{2}} \quad \frac{-3}{2}\cdot\mu + \frac{3}{2}\cdot\lambda + \frac{1}{2}\cdot\left[(-15)\cdot\mu^2 + 18\cdot\mu\cdot\lambda - 3\right.$$

$$\frac{\cdot\mu^3 - 9\cdot\mu\cdot\lambda^2)}{} \quad \frac{-3}{2}\cdot\mu + \frac{3}{2}\cdot\lambda + \frac{1}{2}\cdot\left[(-15)\cdot\mu^2 + 18\cdot\mu\cdot\lambda - 3\cdot\lambda^2\right]^{\frac{1}{2}} \quad \frac{-3}{2}\cdot\mu + \frac{3}{2}\cdot\lambda - \frac{1}{2}\cdot\left[(-15)\cdot\mu^2 + 18\cdot\mu\cdot\lambda - 3\right.$$

$$\begin{bmatrix} 3\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^{3}\cdot\left[\frac{-4}{5}+\frac{3}{10}\cdot 6^{\frac{1}{2}}+2\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^{2}\right]\cdot\lambda^{2} \\ 3\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^{2}\cdot\left[5\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^{2}-\frac{9}{5}+\frac{4}{5}\cdot 6^{\frac{1}{2}}\right]\cdot\lambda \\ 10\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^{3}-18\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^{2}+\frac{22}{5}-\frac{9}{10}\cdot 6^{\frac{1}{2}} \\ \left(\frac{3}{5}+\frac{3}{20}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda-\frac{1}{2}\cdot\left[(-3)\cdot\lambda^{2}+18\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda^{2}-15\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^{2}\cdot\lambda^{2}\right]^{\frac{1}{2}} \\ \left(\frac{3}{5}+\frac{3}{20}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda+\frac{1}{2}\cdot\left[(-3)\cdot\lambda^{2}+18\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)\cdot\lambda^{2}-15\cdot\left(\frac{3}{5}-\frac{1}{10}\cdot 6^{\frac{1}{2}}\right)^{2}\cdot\lambda^{2}\right]^{\frac{1}{2}} \end{bmatrix}$$

K^2

$$\text{Kd}\cdot s^2\Big)\cdot\frac{K}{J\cdot s^3\cdot L + J\cdot s^2\cdot R + b\cdot L\cdot s^2 + s\cdot b\cdot R + s\cdot K^2 + K\cdot Ki + K\cdot Kp\cdot s + K\cdot Kd\cdot s^2}$$

$$\left|\begin{array}{l}\text{simplify}\\ \text{collect}, s \\ \text{explicit}\end{array}\right. \rightarrow K\cdot\left(Ki_i + Kp_i\cdot s + Kd_i\cdot s^2\right)\cdot\frac{}{s^5\cdot J\cdot L + \left(K\cdot Kd_i + J\cdot R + b\cdot L + K\cdot Kd_o\cdot Kd_i\right)\cdot s^4 + \left(K\cdot Kd_o\cdot Kp_i + \text{I}\right.}$$

i

$$\cdot\lambda^{2}\Big]^{\frac{1}{2}}$$

$$\cdot\lambda^{2}\Big]^{\frac{1}{2}}\Bigg]$$

$$\frac{Ki_o}{\cdot Kp_o \cdot Kd_i + K \cdot Kp_i + b \cdot R + K^2\Big) \cdot s^3 + \Big(K \cdot Ki_o \cdot Kd_i + K \cdot Kp_o \cdot Kp_i + K \cdot Kd_o \cdot Ki_i + K \cdot Ki_i\Big) \cdot s^2 + \Big(K \cdot Ki_o \cdot Kp_i + K}$$

$$\overline{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad}$$
$$\cdot Kp_o \cdot Ki_i)\cdot s + K\cdot Ki_o \cdot Ki_i$$