I-P Control of a FOPDT Plant

$$G_p(s) = \frac{K_p \cdot e^{-\theta_p \cdot s}}{\tau_p \cdot s + 1}$$

FOPDT Plant.

$$G_c(s) = K_c \cdot \left(1 + \frac{1}{\tau_i \cdot s}\right)$$

PI Controller

$$T(s) = \frac{K_c \cdot \left(1 + \frac{1}{\tau_i \cdot s}\right) \cdot \frac{K_p \cdot e^{-\theta_p \cdot s}}{\tau_p \cdot s + 1}}{1 + K_c \cdot \left(1 + \frac{1}{\tau_i \cdot s}\right) \cdot \frac{K_p \cdot e^{-\theta_p \cdot s}}{\tau_p \cdot s + 1}}$$

Closed loop transfer function

$$T(s) = \frac{G_c(s) \cdot G_p(s)}{1 + G_c(s) \cdot G_p(s)}$$

Approximate the dead time $e^{-\theta_p \cdot s}$ using the first order Padé approximation $1 - \theta_p \cdot s$ and simplify so the characteristic equation is in the denominator

$$T(s) = \frac{K_c \cdot (\tau_i \cdot s + 1) \cdot K_p \cdot (1 - \theta_p \cdot s)}{\left[(-K_c) \cdot K_p \cdot \theta_p \cdot \tau_i + \tau_i \cdot \tau_p\right] \cdot s^2 + (\tau_i + K_c \cdot K_p \cdot \tau_i - K_c \cdot K_p \cdot \theta_p) \cdot s + K_c \cdot K_p}$$

$$\left[(-K_c) \cdot K_p \cdot \theta_p \cdot \tau_i + \tau_i \cdot \tau_p\right] \cdot s^2 \ldots + (\tau_i + K_c \cdot K_p \cdot \tau_i - K_c \cdot K_p \cdot \theta_p) \cdot s \ldots + K_c \cdot K_p \quad \text{coeffs}, s \rightarrow \begin{bmatrix} K_c \cdot K_p \\ \tau_i + K_c \cdot K_p \cdot \tau_i - K_c \cdot K_p \cdot \theta_p \\ (-K_c) \cdot K_p \cdot \theta_p \cdot \tau_i + \tau_i \cdot \tau_p \end{bmatrix}$$

The desired response

$$\frac{1 \cdot e^{-\theta_p \cdot s}}{(\lambda \cdot s + 1)^2} \quad \text{is approximated by} \quad \frac{1 \cdot (1 - \theta_p \cdot s)}{(\lambda \cdot s + 1)^2}$$

Desired characteristic equation. The desired response is critically damped with two time constants of λ and a dead time approximation. Two time constants are needed so the highest power of s in the actual and desired characteristic equations are the same.

$$\frac{1 \cdot (1 - \theta_p \cdot s)}{(\lambda \cdot s + 1)^2} \,\middle|\, \begin{matrix}\text{expand} \\ \text{collect}, s\end{matrix} \rightarrow \frac{1 - \theta_p \cdot s}{\lambda^2 \cdot s^2 + 2 \cdot \lambda \cdot s + 1}$$

The desired characteristic equation

$$\lambda^2 \cdot s^2 + 2 \cdot \lambda \cdot s + 1 \text{ coeffs}, s \rightarrow \begin{pmatrix} 1 \\ 2 \cdot \lambda \\ \lambda^2 \end{pmatrix}$$

Given

$$\lambda^2 = \frac{(-K_c)\cdot K_p\cdot\theta_p\cdot\tau_i + \tau_i\cdot\tau_p}{K_c\cdot K_p}$$

Equate the coefficients for each of the powers of s and solve for K_c and τ_i.

$$2\cdot\lambda = \frac{\tau_i + K_c\cdot K_p\cdot\tau_i - K_c\cdot K_p\cdot\theta_p}{K_c\cdot K_p}$$

$$\text{Find}(K_c,\tau_i) \rightarrow \begin{bmatrix} \dfrac{-\left(\lambda^2 - 2\cdot\tau_p\cdot\lambda - \tau_p\cdot\theta_p\right)}{\left(2\cdot\lambda\cdot\theta_p + \lambda^2 + {\theta_p}^2\right)\cdot K_p} \\ \dfrac{-\left(\lambda^2 - 2\cdot\tau_p\cdot\lambda - \tau_p\cdot\theta_p\right)}{\theta_p + \tau_p} \end{bmatrix}$$

Formulas for the controller gain K_c
and
controller integrator time constant τ_i.

I find it interesting that the controller gain and integrator time constant have the same numerator.

Define the plant parameters

$K_p := 1$ $\qquad$ $\tau_p := 1$ $\qquad$ $\theta_p := 3$

Set the closed loop time constant.

λ must be less than $\tau_p + \sqrt{\left({\tau_p}^2 + \tau_p\cdot\theta_p\right)} = 3$ to avoid calculating negative controller gains

$\lambda_{max} := \tau_p + \sqrt{{\tau_p}^2 + \tau_p\cdot\theta_p}$ $\qquad$ $\lambda_{max} = 3$

$\lambda := \min\left(\max\left(0.1\cdot\tau_p,\theta_p\right), 0.95\cdot\lambda_{max}\right)$ $\qquad$ $\lambda = 2.85$

Calculate the controller gain and integrator time constant.

$$K_c := \frac{2\cdot\tau_p\cdot\lambda + \tau_p\cdot\theta_p - \lambda^2}{\left(2\cdot\lambda\cdot\theta_p + \lambda^2 + {\theta_p}^2\right)\cdot K_p} \qquad K_c = 0.017$$

$$\tau_i := \frac{2\cdot\tau_p\cdot\lambda + \tau_p\cdot\theta_p - \lambda^2}{\theta_p + \tau_p} \qquad \tau_i = 0.144$$

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Calculate the coefficients for simulating the plant

$T := \frac{1}{600}$ Controller update period in minutes

$A := \exp\left(\frac{-T}{\tau_p}\right)$ $A = 0.998$

$B := K_p \cdot (1 - A)$ $B = 0.002$

Initial conditions.

$PV_{ss} := 25$ Steady state or ambient temperature in degrees centigrade

$PV_0 := PV_{ss}$ Start at ambient temperature

$CO_0 := 0$ Start with no control output

$n\theta_p := \text{floor}\left(\frac{\theta_p}{T}\right)$ $n\theta_p = 1800$ Dead time in update control periods

The PI controller and simulation

$$CL(sp, pv, u, u_\theta) := \begin{array}{|l} pv_{last} \leftarrow pv \\ pv \leftarrow A \cdot (pv - PV_{ss}) + B \cdot u_\theta + PV_{ss} \\ u \leftarrow \max\left[\min\left[u + K_c \cdot \left[\frac{T}{\tau_i} \cdot (sp - pv) - (pv - pv_{last})\right], 100\right], 0\right] \\ \begin{pmatrix} pv \\ u \end{pmatrix} \end{array}$$

$N := \frac{30 \cdot \lambda}{T}$ $n := 0..N$ Simulation periods

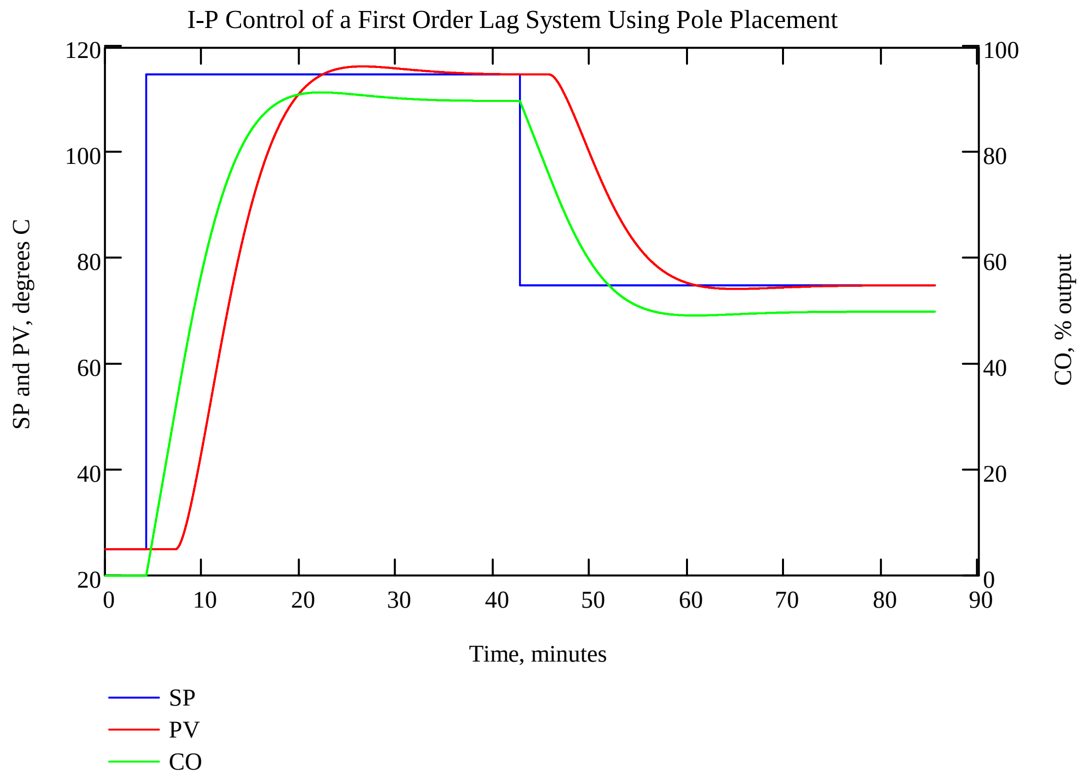
$$SP_n := \begin{array}{|l} PV_{ss} \text{ if } n \le 0.05 \cdot N \\ PV_{ss} + 90 \text{ if } 0.05 \cdot N \le n \wedge n < 0.5 \cdot N \\ PV_{ss} + 50 \text{ otherwise} \end{array}$$

Reference or SP as a function of time period n. Each period is T seconds long.

Calculate the PV and CO for each time period in the simulation. The CO delayed by dead time must be used to simulate the changes in the PV

$$\begin{pmatrix} PV_{n+1} \\ CO_{n+1} \end{pmatrix} := CL\left(SP_n, PV_n, CO_n, \text{if}\left(n \ge n\theta_p, CO_{n-n\theta_p}, CO_0\right)\right)$$

I-P Control of a FOPDT Plant



The PI controller and simulation for IMC gains

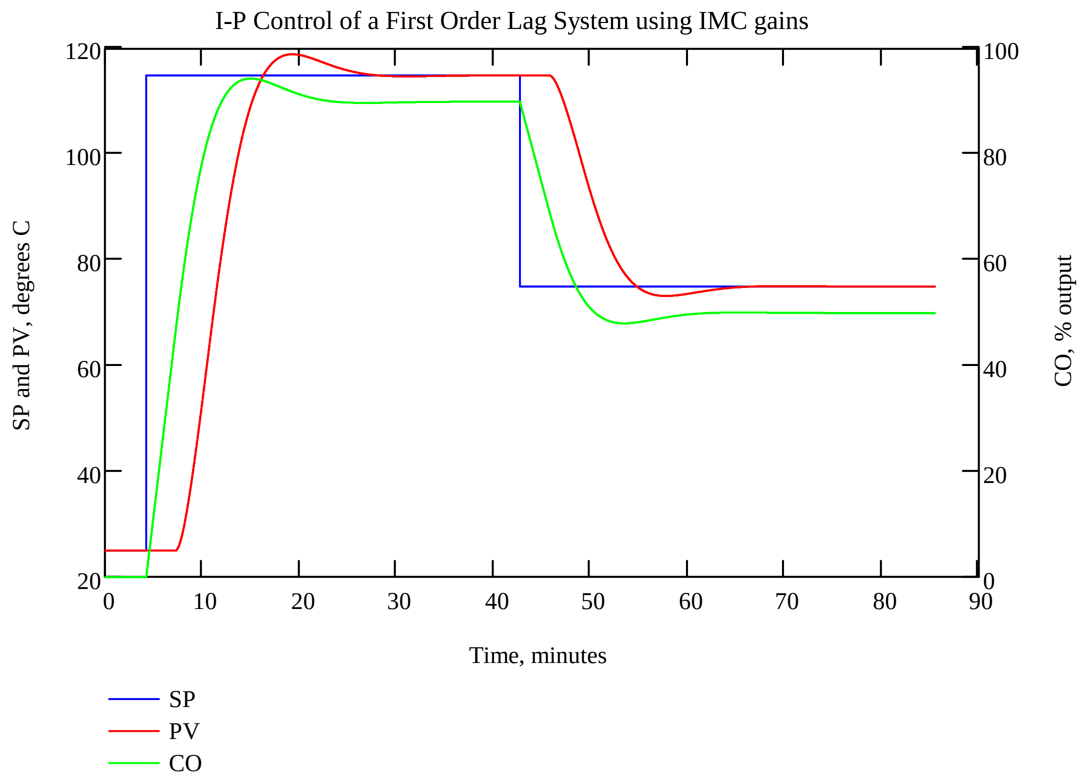
$$K_c := \frac{\tau_p}{K_p \cdot (\lambda + \theta_p)} \qquad K_c = 0.171 \qquad \theta_p = 3$$

$$\tau_i := \tau_p \qquad \tau_i = 1$$

$$CL(sp, pv, u, u_\theta) := \left| \begin{array}{l} pv_{last} \leftarrow pv \\ pv \leftarrow A \cdot (pv - PV_{ss}) + B \cdot u_\theta + PV_{ss} \\ u \leftarrow \max\left[\min\left[u + K_c \cdot \left[\frac{T}{\tau_i} \cdot (sp - pv) - (pv - pv_{last})\right], 100\right], 0\right] \\ \begin{pmatrix} pv \\ u \end{pmatrix} \end{array} \right.$$

Calculate the PV and CO for each time period in the simulation. The CO delayed by dead time must be used to simulate the changes in the PV

$$\begin{pmatrix} PV_{n+1} \\ CO_{n+1} \end{pmatrix} := CL\left(SP_n, PV_n, CO_n, \text{if}\left(n \geq n\theta_p, CO_{n-n\theta_p}, CO_0\right)\right)$$



The IMC gains are a little faster because the desired response is a one pole system instead of a two pole system as in the pole placement example. For the 'standard' plant parameters of K_p=1, τ_p=1, and θ_p=1 the IMC and pole placement gains are identical if λ=1